MATH 3F03 Assignment 2

Solutions

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1. pg 57, ex. 2

i. a. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$

$\Rightarrow \lambda_1 = -1, \; \lambda_2 = 1$

b. for $\lambda_1 = -1$, find eigenvector $V_1 = \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}$

$A + I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ so $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \overset{\text{row-reduction}}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow V_{21} = -V_{11}$

Choose $V_{11} = 1 \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

for $\lambda_2 = 1$, find eigenvector $V_2 = \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix}$

$A - I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ so $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow V_{22} = V_{12}$

Choose $V_{12} = 1 \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So the matrix $T = (V_1 \; V_2) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ puts A into the canonical form $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ via the similarity relation $J = T^{-1}AT$

c. The general solution to $\bar{X}' = A\bar{X}$ is given by

$\bar{X}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (see theorem on pg. 35)

and to $\bar{Y}' = (T^{-1}AT)\bar{Y} = J\bar{Y}$ by

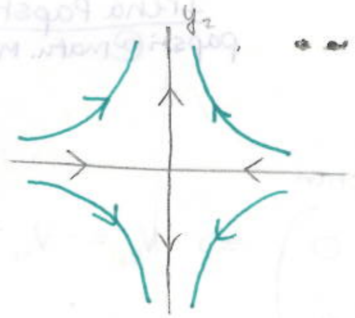
$\bar{Y}(t) = c_3 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_4 e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (by the same theorem)

Note that these can be seen to be the eigenvectors of J, which is a diagonal matrix since the (trivial) similarity relation

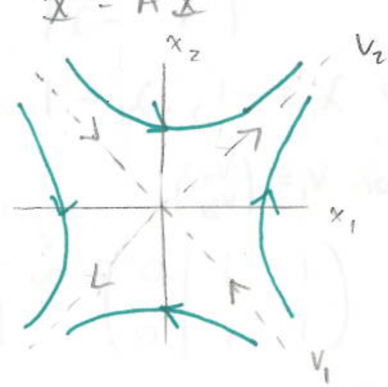
$J = I^{-1} J I$ makes it similar to a diagonal matrix (itself).

d. phase portraits for

$\underline{Y}' = J\underline{Y}$



$\underline{X}' = A\underline{X}$



saddle

vi. a. $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 1$

$$= -1 + \lambda^2 - 1$$

$$= \lambda^2 - 2 = (\lambda - \sqrt{2})(\lambda + \sqrt{2})$$

so $\lambda_1 = -\sqrt{2}$, $\lambda_2 = \sqrt{2}$.

b. $\underline{\text{for } \lambda_1 = -\sqrt{2}}$, eigenvector $V_1 = \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}$

$A + \sqrt{2} I = \begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix}$ so $\left(\begin{array}{cc|c} 1+\sqrt{2} & 1 & 0 \\ 1 & -1+\sqrt{2} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1+\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow (1+\sqrt{2}) V_{11} = -V_{21}$.

Choose $V_{21} = -1 \Rightarrow V_{11} = \dfrac{1}{1+\sqrt{2}}$, so $V_1 = \begin{pmatrix} \dfrac{1}{1+\sqrt{2}} \\ -1 \end{pmatrix}$

$\underline{\text{for } \lambda_2 = \sqrt{2}}$, eigenvector $V_2 = \begin{pmatrix} V_{12} \\ V_{22} \end{pmatrix}$

$A - \sqrt{2} I = \begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}$ so $\left(\begin{array}{cc|c} 1-\sqrt{2} & 1 & 0 \\ 1 & -1-\sqrt{2} & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1-\sqrt{2} & 1 & 0 \\ 1 & -1-\sqrt{2} & 0 \end{array}\right)$

$\Rightarrow (1-\sqrt{2}) V_{12} = -V_{22}$

Choose $V_{22} = -1 \Rightarrow V_{12} = \dfrac{1}{1-\sqrt{2}}$ so $V_2 = \begin{pmatrix} \dfrac{1}{1-\sqrt{2}} \\ -1 \end{pmatrix}$

so the matrix $T = \begin{pmatrix} \dfrac{1}{1+\sqrt{2}} & \dfrac{1}{1-\sqrt{2}} \\ -1 & -1 \end{pmatrix}$ puts A into the canonical form

$$J = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

c. The general solution $\underline{X}' = A\underline{X}$ is given by

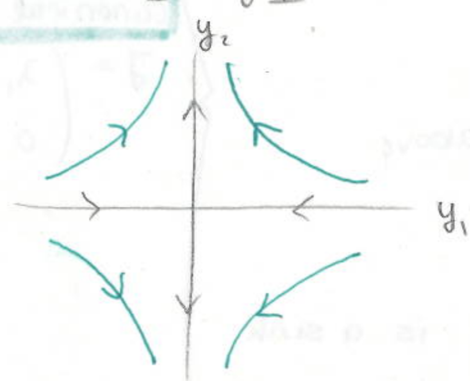
$$\underline{X}(t) = c_1 e^{-\sqrt{2}t}\begin{pmatrix} \frac{1}{1+\sqrt{2}} \\ -1 \end{pmatrix} + c_2 e^{\sqrt{2}t}\begin{pmatrix} \frac{1}{1-\sqrt{2}} \\ -1 \end{pmatrix}$$

and to $\underline{Y}' = J\underline{Y}$ by

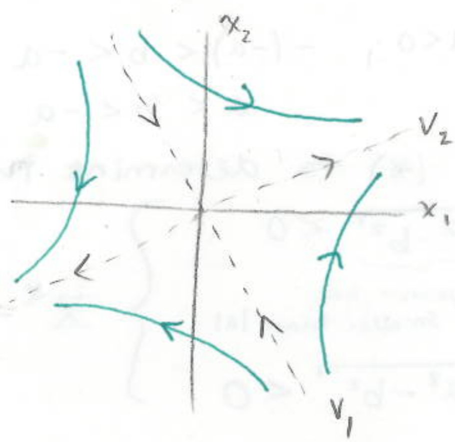
$$\underline{Y}(t) = c_3 e^{-\sqrt{2}t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_4 e^{\sqrt{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

d. phase portraits for

$\underline{Y}' = J\underline{Y}$



$\underline{X}' = A\underline{X}$



2. pg. 58, ex 5

This question was solved in tutorial... if you missed it, ask a classmate for notes.

3. pg. 58, ex 6

$$\underline{X}' = \underbrace{\begin{pmatrix} 2a & b \\ b & 0 \end{pmatrix}}_{A}\underline{X}$$

We want to find the regions in the ab-plane where this system has different canonical forms ... so consider the eigenvalues/vectors of A...

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4\det(A)}}{2} = \frac{2a \pm \sqrt{4a^2+4b^2}}{2} = \frac{\cancel{2}a \pm \cancel{2}\sqrt{a^2+b^2}}{\cancel{2}}$$

$$\Rightarrow \underbrace{\lambda_{1,2} = a \pm \sqrt{a^2+b^2}}$$

Now let's break this up into cases for different types of $\underline{\text{canonical forms}}$

Note that $a^2+b^2 \geq 0$ for all a, b, so we will never have complex eigenvalues

Case ①: λ_1, λ_2 are real & distinct

$\Rightarrow a^2+b^2>0$ & $a + \sqrt{a^2+b^2} \neq a - \sqrt{a^2+b^2}$

↳ true as long as a & b are both nonzero

↳ also always satisfied as long as a & b are both non-zero

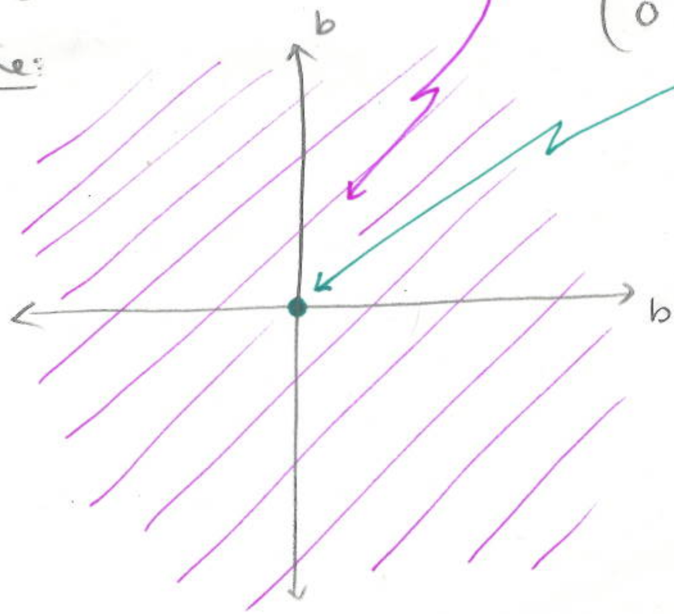
So the corresponding $\boxed{\text{canonical form}}$ is $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Case ②: λ is repeated

$\Rightarrow a^2+b^2=0 \iff a=0$ & $b=0$.

Then, the original matrix A becomes $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ } $\boxed{\text{canonical form}}$ where every point is an equilibrium

The ab-plane:



Note that the question only asks you to sketch the regions in the ab-plane with different canonical forms not phase portraits, so the sketch above is sufficiently detailed.

4. pg 59, ex 11.

$\mathbb{X}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbb{X}$ } You can solve this system using eigenvalue/eigenvector analysis, but let's take a more direct approach...

Let $\mathbb{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then the matrix equation can be broken up into the following two equations:

& $\mathbb{X}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

① $x' = y$

② $y' = 0 \Rightarrow y(t) = c_1$, where c_1 is some constant of integration

Letting $t=0$, $y(0) = c = y_0$, so $y(t) = y_0$ for all t.

Having solved equation ②, we can use this result in ①:

$x' = y_0 \Rightarrow x(t) = y_0 t + c_2$ where c_2 is some constant of integration

Letting $t=0$, $x(0) = y_0(0) + c_2 = x_0 \Rightarrow c_2 = x_0$, so $x(t) = y_0 t + x_0$.

So, what do these solutions tell us? Well, $y(t)$ is constant for all time, so the y-coordinates of our trajectories never change.

$\Rightarrow$ our trajectories are horizontal lines.

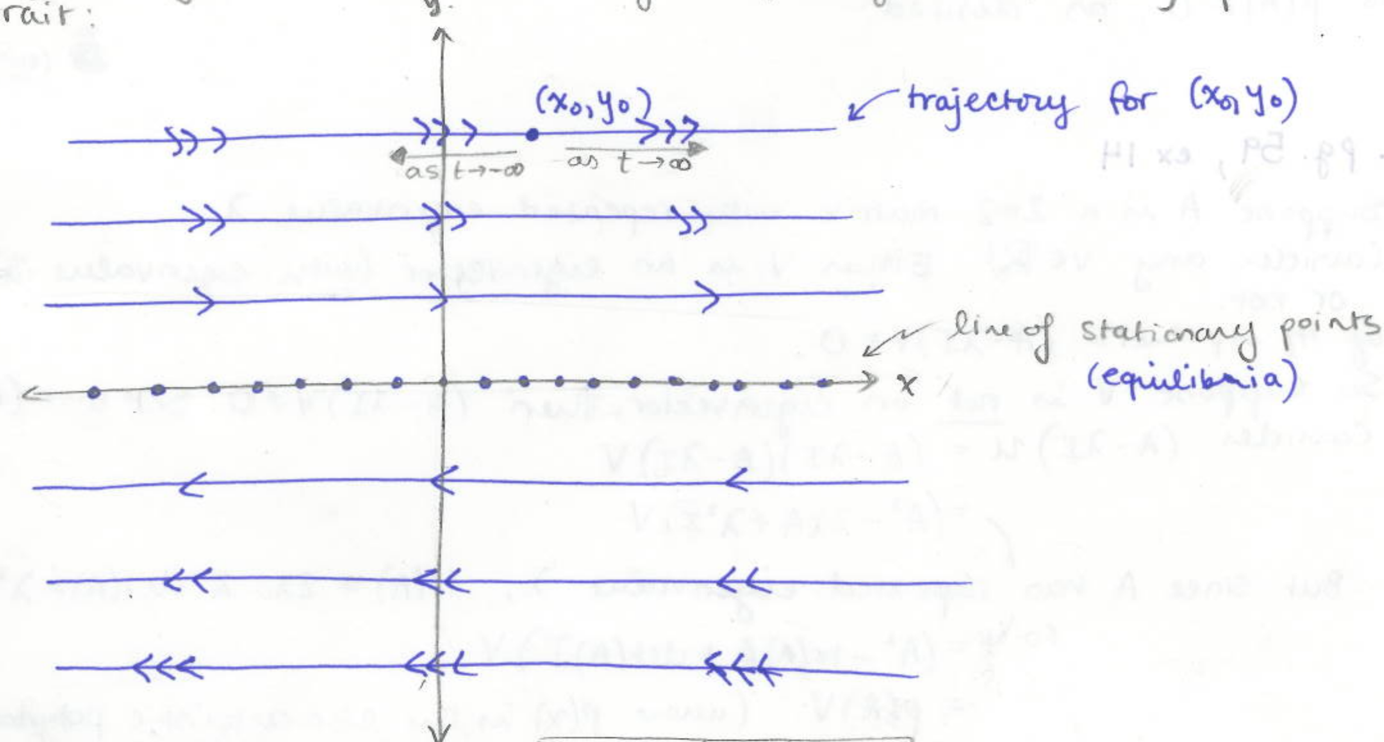
How do we move along these trajectories? Look at $x(t)$.

$x(t) = y_0 t + x_0$, so at $t=0$, we are at (x_0, y_0). (by definition, too)

As $t \to \infty$, $x(t) \to \begin{cases} +\infty & \text{if } y_0 > 0 \\ -\infty & \text{if } y_0 < 0 \end{cases}$ & if $y_0 = 0$, $x(t) = x_0 \to$ constant

Note that the larger $|y_0|$ is, the faster we move along the trajectory.

Putting all of this information together, we get the following phase portrait:



The general solution can be written in matrix form as: $\boxed{\mathbb{X}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mathbb{X}_0}$

5. pg. 59, ex 13

We want to show that a 2×2 matrix A always satisfies its own characteristic equation.
(Note that this is actually the statement of the Cayley-Hamilton theorem for a 2×2 matrix. While this theorem holds for n×n matrices, we don't have to show that here. We'll use the fact that A is 2×2 to write a direct proof.)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$
$= ad - (a+d) + \lambda^2 - b$
$= \lambda^2 - (a+d)\lambda^2 + ad - b$

So $p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$

Consider $p(A) = A^2 - tr(A)A^1 + \det(A)A^0$

$$= A^2 - tr(A)A + \det(A) I$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} + \begin{pmatrix} -a^2-ad & -ab-bd \\ -ac-cd & -ad-d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{a^2}+\cancel{bc}-\cancel{a^2}-\cancel{ad}+\cancel{ad}-\cancel{bc} & \cancel{ab}+\cancel{bd}-\cancel{ab}-\cancel{bd}+0 \\ \cancel{ac}+\cancel{cd}-\cancel{ac}-\cancel{cd}+0 & \cancel{bc}+\cancel{d^2}-\cancel{ad}-\cancel{d^2}+\cancel{ad}-\cancel{bc} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So $p(A) = 0$, as desired.

(end of proof ⇒ coffeetime!)

6. pg. 59, ex 14

Suppose A is a 2×2 matrix with repeated eigenvalue λ.
Consider any $v \in \mathbb{R}^2$. Either v is an eigenvector (with eigenvalue λ) or not.
If it is, then $(A-\lambda I)v = 0$.
So suppose v is <u>not</u> an eigenvector. Then $(A-\lambda I)v \neq 0$. Set $u := (A-\lambda I)v$.
Consider $(A-\lambda I)u = (A-\lambda I)(A-\lambda I)v$

$$= (A^2 - 2\lambda A + \lambda^2 I)v$$

But since A has repeated eigenvalue λ, $tr(A) = 2\lambda$ & $\det(A) = \lambda^2$

so $= (A^2 - tr(A)A + \det(A)I)v$

$= p(A)v$ (where $p(x)$ is the characteristic polynomial of A)

$= 0$ (since $p(A) = 0$ by previous problem)

Thus $(A-\lambda I)u = 0 \Rightarrow u = \underset{(\neq 0)}{(A-\lambda I)v}$ is an eigenvector for A