MATH 3F03 Assignment 3
Marking Comments

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Preamble

How to use this document: Check your marked assignment solutions for boxed numbers. Each boxed number corresponds to a comment in this document. If any points are deducted from your solution related to a comment, the number of points deducted is included in brackets after that comment.

For example,

7. *Page 555, exercise 6.*

$\boxed{1}$ Did not check stability of equilibria **(-2)**

means that in question 7 (which is exercise 6 on page 555 of *Hirsch, Smale, Devaney*), you forgot to check the stability of equilibria, and so I deducted two points.

All questions are marked out of 10.

Note: Only the most commonly applicable comments are included here. Your assignment may have some additional, specific comments.

Comments

1. *Page 103, exercise 3.*

$\boxed{1}$ Missed the condition $a = 0$ and/or $c = 0$ for real and/or repeated eigenvalues. **(-1)**

3. *Page 104, exercise 6.*

$\boxed{1}$ First, note that in an $n \times n$ matrix, the characteristic polynomial is of degree n. In this example, we have a 5×5 matrix, and so it has a characteristic polynomial of degree 5. We know that the matrix in question has three eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 1 + i$, and $\lambda_3 = 1 - i$. Their algebraic multiplicities[1] must sum to 5. We also know that complex eigenvalues always come in conjugate pairs, so their algebraic multiplicities must sum to an even number.

[1]For an eigenvalue λ, the algebraic multiplicity is the degree to which the term $(x - \lambda)$ appears in the characteristic polynomial $p(x)$. For instance, if $p(x) = (x - \lambda)(x - \mu)^2$, the algebraic multiplicity of λ is 1 and of μ is 2.

Putting this information together, we know that the sum of the algebraic multiplicities of λ_2 and λ_3 must be even and cannot exceed 5.

Thus, there are two possibilities:

i. $\text{alg}(\lambda_2) + \text{alg}(\lambda_3) = 2$,

ii. $\text{alg}(\lambda_2) + \text{alg}(\lambda_3) = 4$.

Case i. implies that $\text{alg}(\lambda_2) = \text{alg}(\lambda_3) = 1$, so each complex eigenvalue appears only once. Furthermore, since each eigenvalue must be matched with at least one eigenvector ($\text{geom}(\lambda) \geq 1$), but cannot have more eigenvectors than its algebraic multiplicity ($\text{geom}(\lambda) \leq \text{alg}(\lambda)$), we can conclude that $\text{geom}(\lambda_2) = 1$ and $\text{geom}(\lambda_3) = 1$. Thus, the corresponding Jordan block would be

$$J_{2,3} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

However, in case ii., we have that $\text{alg}(\lambda_2) = \text{alg}(\lambda_3) = 2$, so we have two possibilities:

a. $\text{geom}(\lambda_2) = \text{geom}(\lambda_3) = 1$,

b. $\text{geom}(\lambda_2) = \text{geom}(\lambda_3) = 2$.

In case ii. a., the corresponding Jordan block would be

$$J_{2,3} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

In case ii. b., the corresponding Jordan block would be

$$J_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

$\boxed{2}$ You forgot to consider case ii.[2] altogether. **(-3)**

$\boxed{3}$ You did not justify your answer. **(-2)**

$\boxed{4}$ It is not true that if an eigenvalue $\lambda \in \mathbb{R}$ is repeated (say, twice – so the algebraic multiplicity of λ is 2) that its Jordan block must take the form:

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

It only takes this form if the geometric multiplicity of λ is 1. But remember that the geometric multiplicity of an eigenvalue is at least 1 and at most equal to the algebraic multiplicity. Thus, the geometric multiplicity of λ could be 2 in this case, and so there is another possible Jordan block:

$$J_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

[2]See $\boxed{1}$.

Analogously, for a complex eigenvalue $\lambda_+ = \alpha + \beta i$, it must have a conjugate pair $\lambda_- = \alpha - \beta i$. If each complex eigenvalue is repeated twice (so the algebraic multiplicity of each is 2), then the geometric multiplicity of each could be 1, and so the Jordan block would be

$$J_{\lambda_\pm} = \begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix},$$

but if the geometric multiplicity of each complex eigenvalue is 2, then the Jordan block would be

$$J_{\lambda_\pm} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}.$$

$\boxed{5}$ You did not account for the cases where the geometric multiplicity of each repeated eigenvalue was 2.[3] **(-3)**

4. *Page 104, exercise 13.*

$\boxed{1}$ Just because a set is not open, does not mean it's closed (and vice-versa), despite what the semantics would lead you to believe. *The topological properties "open" and "closed" are **not** mutually exclusive, nor are they exhaustive.* There exist sets that are only open, only closed, both, and neither.

For instance, take the real line with the familiar notions of open and closed. That is, the interval $(0,1)$ is open, while $[0,1]$ is closed. However, the interval $(0,1]$ is neither open nor closed. This should be familiar.

To see how there can exists sets that are both open and closed on the real line, we have to appeal to the topological definitions of these terms. Let's stick to the real line (and so these definitions will be specific to $\mathbb{R}$, though they can be stated more generally). A set $O \subseteq \mathbb{R}$ is open if, for any point in O there exists an open interval, I, that contains this point such that $I \subset O$. A set $C \subseteq \mathbb{R}$ is closed if the complement of C in $\mathbb{R}$, that is, $\mathbb{R} \setminus C$, is open.

The sets $\emptyset$ (the empty set) and $\mathbb{R}$ are each both open and closed in $\mathbb{R}$. The empty set is open since it does not contain any points and so it (vacuously) satisfies the defintion above. The real line is open since for any point on the line, x_0, we can take, say, the open interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$ and I will always be a subset of $\mathbb{R}$ that contains the point x_0. But $\emptyset$ and $\mathbb{R}$ are both closed since their complements ($\mathbb{R}$ and $\emptyset$, respectively) are open. Thus, we have found two subsets of the real line that are *both* open and closed, and so I hope you are now convinced that "open" and "closed" are not opposites in topology as they may be in real life.

$\boxed{2}$ For part e., the set U_5 could either be dense or not dense – we don't have enough information (see assignment solutions). **(-0.5)**

[3] See $\boxed{4}$.

$\boxed{3}$ Recall that the definition of an open set is as follows:

A set $A \subseteq U$ is *open* if for any point $x \in A$, we can find an open set $B \subset U$ such that:

i. $x \in B$,

ii. $B \subset A$.

$\boxed{4}$ Recall that the definition of a set being dense in another set is as follows:

A set $D \subseteq U$ is *dense* in U if for any point $u \in U \setminus D$ (the complement of D in U), any open set containing y intersects D. That is, if $B \subseteq U$ is any open set such that $y \in B$, then $B \cap U \neq \emptyset$.

Note that a set is not dense on its own, but rather dense relative to a superset. For instance, the set $U_1 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is not dense as a subset of $\mathbb{R}^2$ but it is dense as a subset of $U = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.

$\boxed{5}$ You did not provide any justification for your answers. **(-4)**

7. *Page 137, exercise 12.*

$\boxed{1}$ See the assignment solutions for a faster way of solving parts f. and i. (which could come in handy in a timed testing situation...).

$\boxed{2}$ It is ***not true*** that the exponential of a matrix is simply taking the exponential of each entry in the matrix. For instance, given the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have that

$$e^A \neq \begin{pmatrix} 1 & e \\ 1 & 1 \end{pmatrix}, \tag{1}$$

i.e., the exponential of the matrix A is ***not given by*** (1). To compute the matrix exponential, we have to use the series definition of the exponential function, *i.e.*,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

In other words, we have to compute A^k for various $k = 0, 1, 2, ...$ and find a pattern. For A, we note that,

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad ..., \quad A^k = 0 \quad \forall k \geq 2,$$

and so

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{1} \frac{A^k}{k!} = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This example was particularly easy since A is a nilpotent matrix (of degree 2), and so the matrix exponential is a finite series. Thus, computing e^A in this case boils down to summing a finite number of powers of A. **(-1)**

$\boxed{3}$ When asked to compute the matrix exponential of A, that means compute e^A, not e^{tA}, which is not the matrix exponential of A, but instead the general solution to the linear system of differential equations given by $X' = AX$. **(-1)** or **(-3)**, depending.

8. *Page 138, exercise 14.*
Each part of this question was marked out of five points.

$\boxed{1}$ See the assignment solutions for a simpler proof of part a. that does not use the series defintion of e^A.