MATH 3F03
Assignment 3
Solutions
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1. pg 103, ex 3:

$$A = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} \to \text{real, complex, repeated eigenvalues?}$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & a \\ 0 & b-\lambda & 0 \\ c & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} b-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + a \begin{vmatrix} 0 & b-\lambda \\ c & 0 \end{vmatrix}$$

$$p(\lambda) = \lambda^2(b-\lambda) - ac(b-\lambda) = (\lambda^2 - ac)(b-\lambda) = 0$$

when $\lambda_1 = b$

$\lambda^2 = ac \Rightarrow \lambda_{2,3} = \pm\sqrt{ac}$

- Real eigenvalues
  1. $a>0$ & $c>0$
  2. $a<0$ & $c<0$
  3. $a=0$
  4. $c=0$.
- Complex eigenvalues:
  1. $a>0$ & $c<0$
  2. $a<0$ & $c>0$
- Repeated eigenvalues.
  1. $a=0$
  2. $c=0$
  3. $b=\sqrt{ac}$
  4. $b=-\sqrt{ac}$

2. pg 103, ex 5: canonical form

a. $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix}$

$$= \lambda^2(1-\lambda) - (1-\lambda) = (1-\lambda)(\lambda^2-1) = -(\lambda-1)^2(\lambda+1).$$

$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$, where $\text{alg}(\lambda_1) = 2$, $\text{alg}(\lambda_2) = 1$

for $\lambda_1 = 1$: want $V_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow v_{21}$ & v_{31} are free variables

$v_{11} = v_{31}$

Choose $v_{21} = v_{31} = 1$

Then $v_{11} = 1$.

so we get two, linearly independent eigenvectors:

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \ \&\ V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow \text{geom}(\lambda_1) = 2$

& know $\text{geom}(\lambda_2) = 1$

$\to$ Thus, the canonical form for this matrix is

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

d. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$

$= (1-\lambda)(\lambda^2-1) = -(\lambda-1)^2(\lambda+1)$

$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$, where $alg(\lambda_1) = 2$, $alg(\lambda_2) = 1$

for $\lambda_1 = 1$:

want $V_1 = \begin{pmatrix} V_{11} \\ V_{21} \\ V_{31} \end{pmatrix}$ $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow V_{11} = V_{21} = 0$

V_{31} is a free variable

choose $V_{31} = 1$.

so we get one eigenvector $\Rightarrow geom(\lambda_1) = 1$.

& we know that $geom(\lambda_2) = 1$.

Thus, the canonical form for this matrix is $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

g. $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$

$\Rightarrow \lambda_1 = 1$, where $alg(\lambda_1) = 3$.

for $\lambda_1 = 1$:

want $V_1 = \begin{pmatrix} V_{11} \\ V_{21} \\ V_{31} \end{pmatrix}$ $\left(\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow V_{11} = 0$

V_{21} & V_{31} are free variables.

Choose $V_{21} = V_{31} = 1$.

so we get two, linearly independent eigenvectors: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow geom(\lambda_1) = 2$.

Thus, the canonical form for this matrix is

$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

3. pg. 104, ex 6:

5×5 matrix with eigenvalues $\lambda_1 = 2$ & $\lambda_{2,3}\underbrace{1 \pm i}$ $\Rightarrow$ $alg(\lambda_1) + alg(\lambda_2) + alg(\lambda_3) = 5$

Possible canonical forms?

complex eigenvalues only come in conjugate pairs.

Case ① $\lambda_1 = 2$ is repeated with $alg(\lambda_1) = 3$

$\left.\begin{array}{l} \lambda_2 = 1+i \\ \lambda_3 = 1-i \end{array}\right\}$ $\begin{array}{l} alg(\lambda_2) = 1 \\ alg(\lambda_3) = 1 \end{array}$

ⓐ $geom(\lambda_1) = 3 \Rightarrow J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

ⓒ $geom(\lambda_1) = 1$

$\Rightarrow J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

ⓑ $geom(\lambda_1) = 2 \Rightarrow J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

* Jordan blocks indicated by green boxes.

Case ② $\left.\begin{array}{l} \lambda_2 = 1+i \\ \lambda_3 = 1-i \end{array}\right\}$ are each repeated with $alg(\lambda_2) = alg(\lambda_3) = 2$

$alg(\lambda_1) = 1$

ⓐ $geom(\lambda_2) = geom(\lambda_3) = 2$.

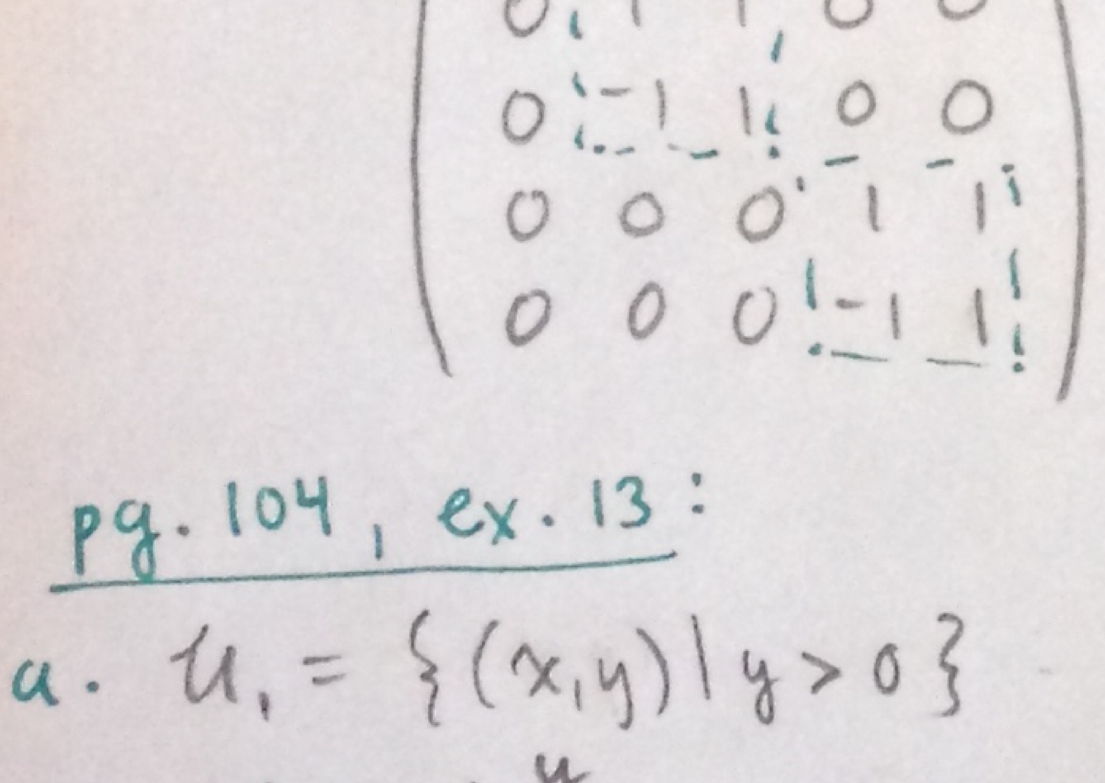
$\Rightarrow J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

ⓑ $geom(\lambda_2) = geom(\lambda_3) = 1$

$\Rightarrow J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

4. pg. 104, ex. 13:

a. $U_1 = \{(x,y) \mid y > 0\}$



minimal distance to boundary (call it r)

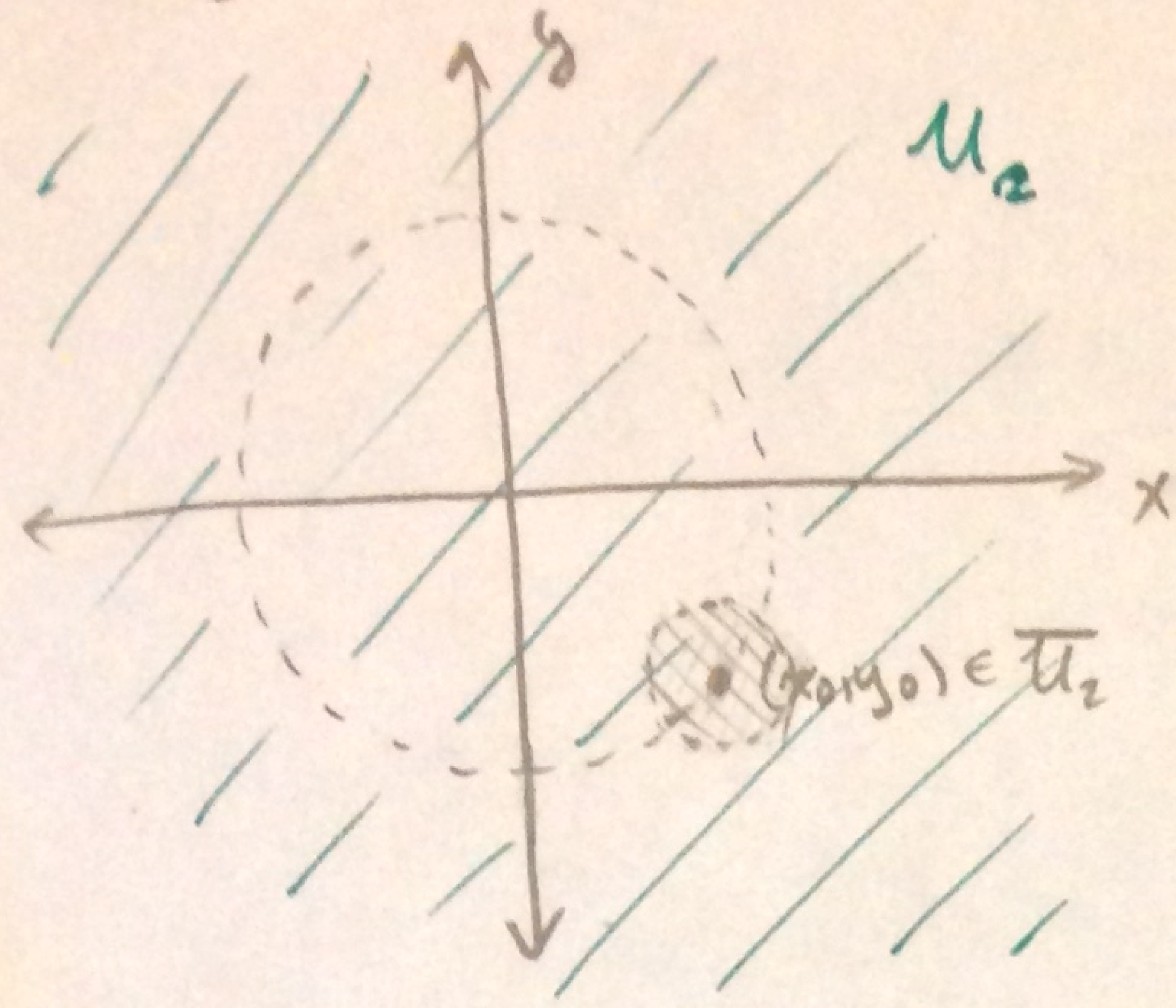
So take a ball of radius $\frac{r}{2}$ about (x_0,y_0)

→ open, since for any point $(x_0,y_0) \in U_1$, we can always find an open ball centred about (x_0,y_0) that is fully contained in U_1 (see diagram)

→ not dense, because a point in the set $V_1 = \{(x,y) \mid y < 0\} \in \mathbb{R}^2$ is not arbitrarily close to U_1 (take $(x,y) = (0,-1)$ … not true that every neighbourhood of $(0,1)$ intersects U_1 → any ball of radius < 1 will not intersect)

"neighbourhood of (x_0,y_0)" = "open ball centred at (x_0,y_0)" in this context

b. $U_2 = \{(x,y) \mid x^2+y^2 \neq 1\}$



→ open, again, because we can always find a neighbourhood around any point in U_2, where the neighbourhood is fully contained in U_2.

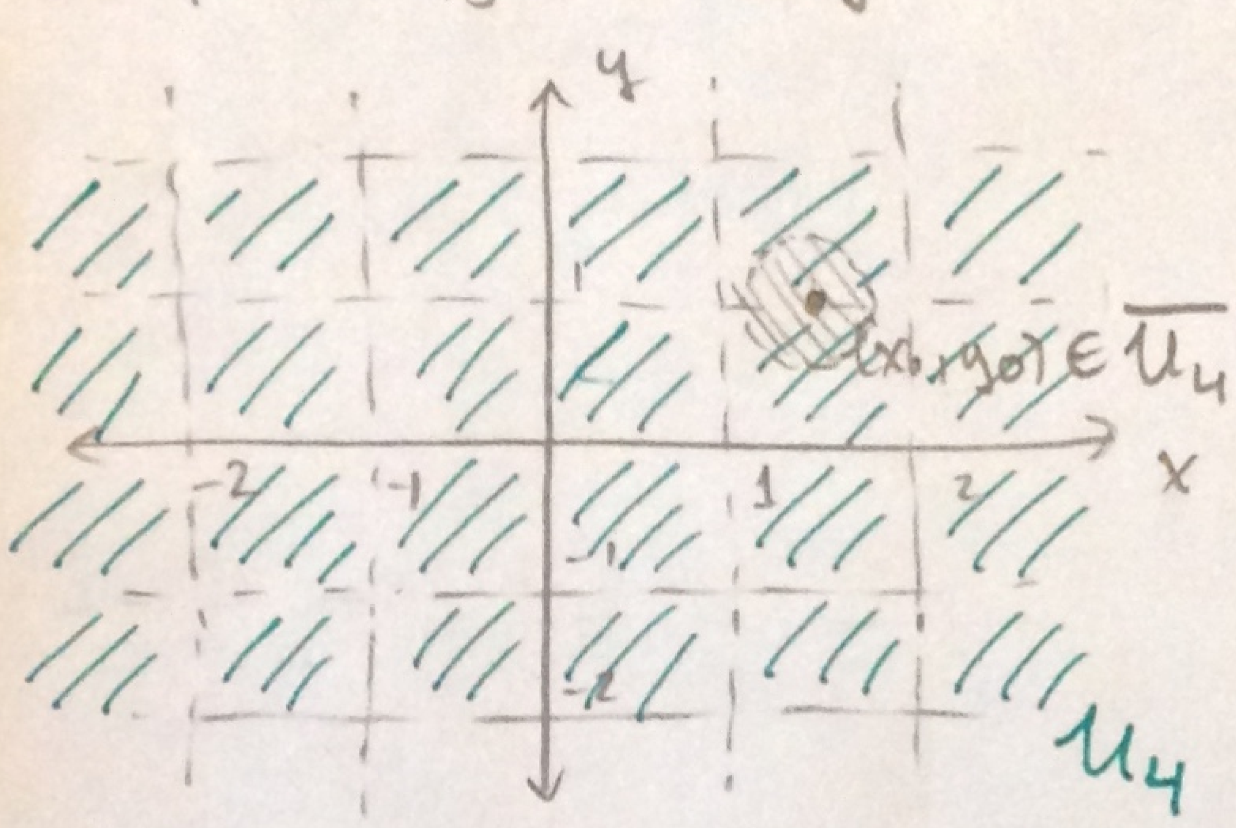
→ dense, because for any point in the complement of $\overline{U_2} = \{(x,y) \mid x^2+y^2 = 1\}$, any neighbourhood of such a point intersects U_2 (see diagram)

c. $U_3 = \{(x,y) \mid x \text{ is irrational}\}$.

→ not open, because any open ball about a point in the set will intersect the complement $\mathbb{R}^2 \setminus U_3$. (every open ball leaves the set).

→ dense, because for any point in the complement, $\mathbb{R}^2 \setminus U_3 = \{(x,y) \mid x \text{ is rational}\}$, an open ball about that point will contain points with a rational x-coordinate.

d. $U_4 = \{(x,y) \mid x \text{ and } y \text{ are not integers}\}$



→ open, because we can take any point in U_4 & find an open ball about that point fully contained in U_4

→ dense, because for any point in the complement $\overline{U_4} = \{(x,y) \mid x \text{ or } y \text{ is an integer}\}$, any open ball about such a point will always intersect U_4 (see diagram)

e. U_5 is the complement of a set C_1 where C_1 is closed & not dense.

→ open because, by definition, a closed set (like C_1) is one where its complement (i.e. U_5 in this case) is open

→ Could either be dense or not... the fact that the complement C_1 is dense is not enough info.

↳ For instance, the complement of U_1 from part a is not dense, and neither is U_1

↳ However, the complement of U_2 from part b is not dense either, but U_2 is.

f. U_6 is the complement of a set C_2 that contains exactly 6 billion and 2 points.

If a subset of $\mathbb{R}^2$ contains a finite number of points (like C_2), then it must be a set of discrete points, for if it contains even one line (continuum), the set will contain infinitely many points.

Thus the complement of U_6 contains finitely many points, which means U_6 is open since, for any point in U_6, we can always find an open set about this point small enough so that it avoids the finitely many points in the complement C_2

$\mathcal{U}_6$ is also dense since any point in C_2, the complement, is a discrete (isolated) point, so any open ball about this point will have to intersect $\mathcal{U}_6$.

5. pg 135, ex 1:

a. $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ → From question 2 of this assignment, we know that A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ & λ_1 has corresponding eigenvectors $V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ & $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = -1$:

want $V_3 = \begin{pmatrix} V_{13} \\ V_{23} \\ V_{33} \end{pmatrix}$: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow V_{23} = 0$

$V_{13} = -V_{33}$ where V_{33} is a free variable.

Choose $V_{33} = -1 \Rightarrow V_{13} = 1$.

so $V_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

So the matrix $T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ puts the matrix A into the canonical form $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ by the relation $A = TJT^{-1}$

we can compute that $T^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$

The general solution to $\mathbf{X}' = A\mathbf{X}$ is given by $\mathbf{X}(t) = e^{tA}\mathbf{X}_0$.

$$\mathbf{X}(t) = e^{tA}\mathbf{X}_0 = e^{t(TJT^{-1})}\mathbf{X}_0 = Te^{tJ}T^{-1}\mathbf{X}_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

which you can multiply out...

c. [redacted] $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ → eigenvalues: $\lambda_1 = 1$, $\lambda_{2,3} = \pm i$

→ eigenvectors: $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}$, $V_3 = \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix}$

$Re(V_2) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $Im(V_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

→ canonical form:

$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $e^{tJ} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & \sin(t) \\ 0 & -\sin(t) & 1 \end{pmatrix}$

→ transformation matrix:

$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, $T^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

→ general solution:

$$\mathbf{X}(t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & \sin(t) \\ 0 & -\sin(t) & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

h. $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ → eigenvalues: $\lambda_1 = 1$, $\lambda_{2,3} = \frac{1 \pm \sqrt{5}}{2}$

→ eigenvectors: $V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\sqrt{5}/2 \end{pmatrix}$, $V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sqrt{5}/2 \end{pmatrix}$

→ generalized eigenvector: $u_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

→ canonical form:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \qquad e^{tJ} = \begin{pmatrix} e^t & t & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{\frac{1+\sqrt{5}}{2}t} & 0 \\ 0 & 0 & 0 & e^{\frac{1-\sqrt{5}}{2}t} \end{pmatrix}$$

→ transformation matrix:

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

→ general solution:

$$X(t) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} e^t & t & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{\frac{1+\sqrt{5}}{2}t} & 0 \\ 0 & 0 & 0 & e^{\frac{1-\sqrt{5}}{2}t} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ u_0 \end{pmatrix}$$

7. pg 137, ex 12

a. $A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$ → eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 2$
→ eigenvectors: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

→ canonical form: $J = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

→ transformation matrix: $T = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $T^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

→ matrix exponential:

$$e^A = e^{TJT^{-1}} = Te^JT^{-1} = \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e^2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}}_{\text{which you can multiply out...}}$$

f. $$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{N}$$

Note that $DN = ND$, so

$$e^A = e^{D+N} = e^De^N$$

Moreover, $N^2 = 0$, so $e^A = e^De^N = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^3 \end{pmatrix} \cdot \sum_{k=0}^{1} \frac{N^k}{k!}$

$$= \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & e^3 & e^3 \end{pmatrix}$$

i. $$A = \begin{pmatrix} 1+i & 0 \\ 2 & 1+i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{R} + i \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$$

Note that $RI = IR$, so

$$e^A = e^{R+iI} = e^R \cdot e^{iI}$$

First consider $R = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \underbrace{\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}}_{N}$ } Again, $IN = NI$ & so $e^R = e^Ie^N$

Now, $N^2 = 0$, so $e^N = \sum_{k=0}^{1} \frac{N^k}{k!} = I + N = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

& so $e^R = e^I e^N = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 2e & e \end{pmatrix}$.

Also, $e^{iI} = \begin{pmatrix} e^i & 0 \\ 0 & e^i \end{pmatrix}$, so $e^A = e^R e^{iI} = \begin{pmatrix} e & 0 \\ 2e & e \end{pmatrix}\begin{pmatrix} e^i & 0 \\ 0 & e^i \end{pmatrix} = \begin{pmatrix} e^{1+i} & 0 \\ 2e^{1+i} & e^{1+i} \end{pmatrix}$

8. pg. 138, ex 14:

Suppose $AB = BA$.

a. Want to show that $e^A e^B = e^B e^A$

start with $\underbrace{e^A e^B = \boxed{e^{A+B}}}_{\text{since } AB=BA} = \underbrace{\boxed{e^{B+A}} = e^B e^A}_{\text{again, since } AB=BA.} \to$ done!

since matrix addition is commutative.

b. Want to show that $e^A B = B e^A$

start with $\underbrace{e^A B = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right) B = \sum_{k=0}^{\infty} \frac{A^k B}{k!}}_{\text{since } B \text{ does not depend on } k\text{, so we can bring it into the series}} = \underbrace{\sum_{k=0}^{\infty} \frac{\overbrace{(A \cdot A \cdots A)}^{k\text{-times}} B}{k!} = \sum_{k=0}^{\infty} \frac{B\overbrace{(A \cdot A \cdots A)}^{k\text{-times}}}{k!}}_{\text{since } AB=BA\text{, we can pass } B \text{ to the left } k\text{-times}}$

$= \sum_{k=0}^{\infty} \frac{BA^k}{k!} = B \sum_{k=0}^{\infty} \frac{A^k}{k!} = Be^A \to$ done!