Solutions

1.) $x' = x^3 - ax = f_a(x)$

$f_a(x) = 0 = x(x^2 - a) \quad (*)$

If $a<0$, $*$ has $x=0$ as the only real solution

If $a=0$, $*$ has $x=0$ as the only solution

If $a>0$, $*$ has $x=0$, $x=\pm\sqrt{a}$ as solutions

Now we analyze the stability of equilibria in each case:

Case I $a<0$, equilibrium $x=0$

$$f_a'(x) = 3x^2 - a$$

$$f_a'(0) = -a > 0 \Rightarrow x=0 \text{ is } \underline{\text{unstable}}$$

Case II $a=0$, equilibrium $x=0$

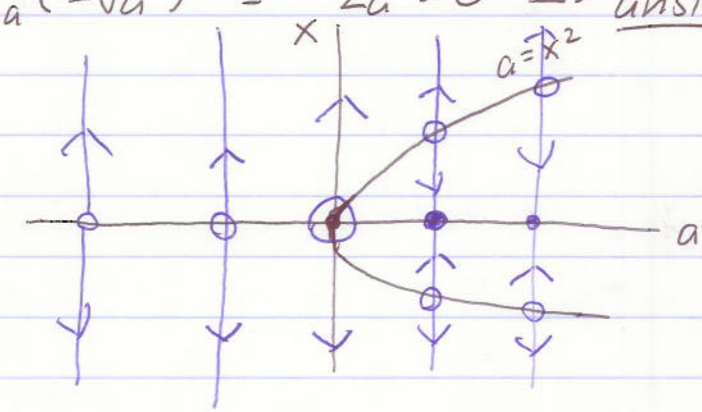
In this case $f_a'(0)=0$, which gives no information. However, if $a=0$, then the equation $x'=x^3$ tells us that $x'>0$ for $x>0$ and $x'<0$ for $x<0$, which means that positive solutions "blow up" to ∞ over time, while negative solutions "blow up" to $-\infty$ over time. In other words, $x=0$ is unstable here.

Case III $a>0$ equilibrium $x=0$

$$f_a'(0) = -a < 0 \Rightarrow \underline{\text{stable}}$$

$$f_a'(+\sqrt{a}) = 2a > 0 \Rightarrow \underline{\text{unstable}}$$

$$f_a'(-\sqrt{a}) = 2a > 0 \Rightarrow \underline{\text{unstable}}$$



2.) ~~Take~~ a.) Set $y = x' \Rightarrow y' = -2x - 3y$

System: $x' = y$, or

$y' = -2x - 3y$

$X' = AX$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

b.) $\det(A - \lambda I) = (\lambda + 2)(\lambda + 1) = 0 \Leftrightarrow \lambda = -1, -2$

eigenvector associated to $\lambda = -1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

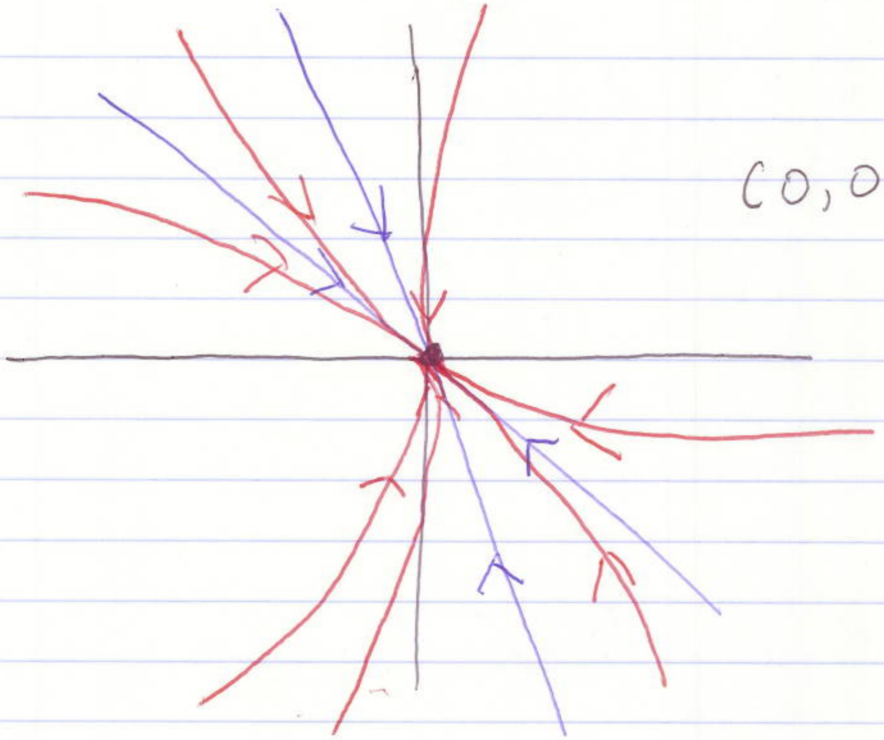
eigenvector associated to $\lambda = -2$ is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Hence the general solution is

$$X(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

c.) $J = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $T = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$

d.)



(0,0) is a

<u>sink</u>

3.) Let $y(t)$ be any other solution to the ODE, so that $y'(t) = a\,y(t)$

Now compute $\frac{d}{dt}\left[y(t)\,e^{-at}\right]$

$$= y'(t)\,e^{-at} - a\,y(t)\,e^{-at} = e^{-at}\left[y'(t) - a\,y(t)\right]$$

$= 0$, because $y' = ay \iff y' - ay = 0$

Hence $y(t)\,e^{-at}$ is constant, so write

$$y(t)\,e^{-at} = k \qquad (k \in \mathbb{R})$$

$$\iff y(t) = ke^{at}$$, as needed

4.) Suppose v_1 and v_2 are <u>not</u> linearly independent. Two vectors are linearly dependent iff one is a scalar multiple of the other; i.e. $\exists\, \alpha \in \mathbb{R} \setminus \{0\}$ such that $v_1 = \alpha v_2$

Apply A to both sides:

$$Av_1 = A(\alpha v_2)$$
$$\lambda_1 v_1 = \alpha \lambda_2 v_2$$
$$\lambda_1 v_1 = \lambda_2 v_1$$
$$(\lambda_1 - \lambda_2)\,v_1 = 0$$

$\implies (\lambda_1 - \lambda_2) = 0$, since v_1 is an eigenvector and hence cannot be $\vec{0}$

But, $\lambda_1 - \lambda_2 = 0 \iff \lambda_1 = \lambda_2$, which is a contradiction

Hence we must have v_1 and v_2 linearly independent.

5.) 2×2 Hyperbolic matrices A_1 and A_2 are conjugate iff they have the same number of eigenvalues with negative real parts. This yields 3 conjugacy classes:

(i) Sources, with real parts of both eigenvalues positive

(ii) Sinks, with real parts of both eigenvalues negative

(iii) Saddles, with real part of one eigenvalue negative, real part of the other eigenvalue positive