tion damps out and the pendulum comes to rest at its stable equilibrium.

This example shows how far we can go with pictures—without invoking any difficult formulas, we were able to extract all the important features of the pendulum's dynamics. It would be much more difficult to obtain these results analytically, and much more confusing to interpret the formulas, even if we *could* find them.

6.8 Index Theory

In Section 6.3 we learned how to linearize a system about a fixed point. Linearization is a prime example of a *local* method: it gives us a detailed microscopic view of the trajectories near a fixed point, but it can't tell us what happens to the trajectories after they leave that tiny neighborhood. Furthermore, if the vector field starts with quadratic or higher-order terms, the linearization tells us nothing.

In this section we discuss index theory, a method that provides *global* information about the phase portrait. It enables us to answer such questions as: Must a closed trajectory always encircle a fixed point? If so, what types of fixed points are permitted? What types of fixed points can coalesce in bifurcations? The method also yields information about the trajectories near higher-order fixed points. Finally, we can sometimes use index arguments to rule out the possibility of closed orbits in certain parts of the phase plane.

The Index of a Closed Curve

The index of a closed curve C is an integer that measures the winding of the vector field on C. The index also provides information about any fixed points that might happen to lie inside the curve, as we'll see.

This idea may remind you of a concept in electrostatics. In that subject, one often introduces a hypothetical closed surface (a "Gaussian surface") to probe a configuration of electric charges. By studying the behavior of the electric field on the surface, one can determine the total amount of charge *inside* the surface. Amazingly, the behavior *on* the surface tells us what's happening far away *inside* the surface! In the present context, the electric field is analogous to our vector field, the Gaussian surface is analogous to the curve C, and the total charge is analogous to the index.

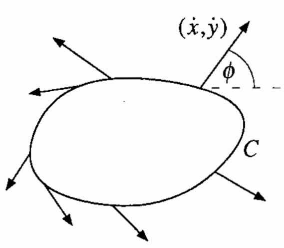

Figure 6.8.1

Now let's make these notions precise. Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a smooth vector field on the phase plane. Consider a closed curve C (Figure 6.8.1). This curve is *not* necessarily a trajectory—it's simply a loop that we're putting in the phase plane to probe the behavior of the vector field. We also assume that C is a "simple closed curve" (i.e., it doesn't intersect itself) and that it doesn't pass through any fixed points of the system. Then at each point $\mathbf{x}$ on C, the vector field $\dot{\mathbf{x}} = (\dot{x}, \dot{y})$ makes a well-defined angle

$$\phi = \tan^{-1}(\dot{y}/\dot{x})$$

with the positive x-axis (Figure 6.8.1).

As $\mathbf{x}$ moves counterclockwise around C, the angle ϕ changes *continuously* since the vector field is smooth. Also, when $\mathbf{x}$ returns to its starting place, ϕ returns to its original direction. Hence, over one circuit, ϕ has changed by an *integer* multiple of 2π. Let $[\phi]_C$ be the net change in ϕ over one circuit. Then the ***index of the closed curve*** C with respect to the vector field $\mathbf{f}$ is defined as

$$I_C = \tfrac{1}{2\pi}[\phi]_C .$$

Thus, I_C is the net number of counterclockwise revolutions made by the vector field as $\mathbf{x}$ moves once counterclockwise around C.

To compute the index, we do not need to know the vector field everywhere; we only need to know it along C. The first two examples illustrate this point.

EXAMPLE 6.8.1:

Given that the vector field varies along C as shown in Figure 6.8.2, find I_C.

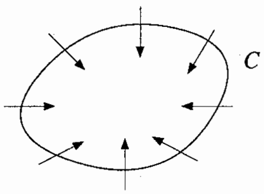

Figure 6.8.2

Solution: As we traverse C once counterclockwise, the vectors rotate through one full turn in the same sense. Hence $I_C = +1$.

If you have trouble visualizing this, here's a foolproof method. Number the vectors in counterclockwise order, starting anywhere on C (Figure 6.8.3a). Then transport these vectors *(without rotation!)* such that their tails lie at a common origin (Figure 6.8.3b). The index equals the net number of counterclockwise revolutions made by the numbered vectors.

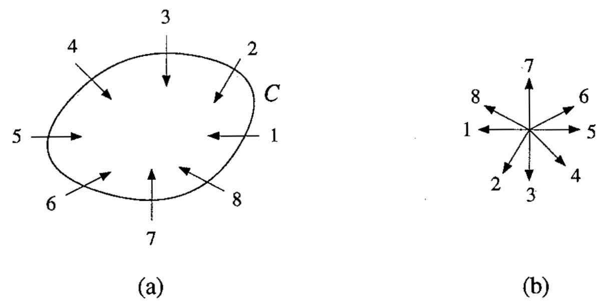

Figure 6.8.3

As Figure 6.8.3b shows, the vectors rotate once counterclockwise as we go in increasing order from vector #1 to vector #8. Hence $I_C = +1$. ■

EXAMPLE 6.8.2:

Given the vector field on the closed curve shown in Figure 6.8.4a, compute I_C.

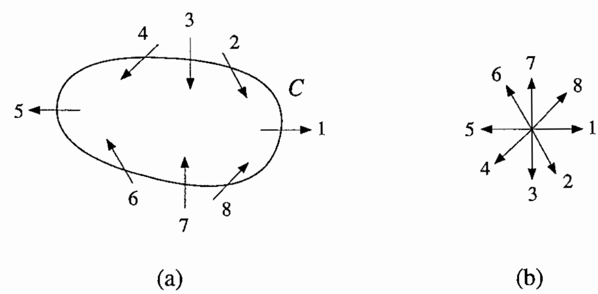

Figure 6.8.4

Solution: We use the same construction as in Example 6.8.1. As we make one circuit around C, the vectors rotate through one full turn, but now in the *opposite* sense. In other words, the vectors on C rotate *clockwise* as we go around C counterclockwise. This is clear from Figure 6.8.4b; the vectors rotate clockwise as we go in increasing order from vector #1 to vector #8. Therefore $I_C = -1$. ■

In many cases, we are given equations for the vector field, rather than a picture of it. Then we have to draw the picture ourselves, and repeat the steps above. Sometimes this can be confusing, as in the next example.

EXAMPLE 6.8.3:

Given the vector field $\dot{x} = x^2 y$, $\dot{y} = x^2 - y^2$, find I_C, where C is the unit circle $x^2 + y^2 = 1$.

Solution: To get a clear picture of the vector field, it is sufficient to consider a few conveniently chosen points on C. For instance, at $(x, y) = (1,0)$, the vector is $(\dot{x}, \dot{y}) = (x^2 y,\ x^2 - y^2) = (0,1)$. This vector is labeled #1 in Figure 6.8.5a. Now we move counterclockwise around C, computing vectors as we go. At $(x, y) = \frac{1}{\sqrt{2}}(1,1)$, we have $(\dot{x}, \dot{y}) = (x, y) = \frac{1}{2\sqrt{2}}(1,0)$, labeled #2. The remaining vectors are found similarly. Notice that different points on the circle may be associated with the same vector; for example, vector #3 and #7 are both $(0,-1)$.

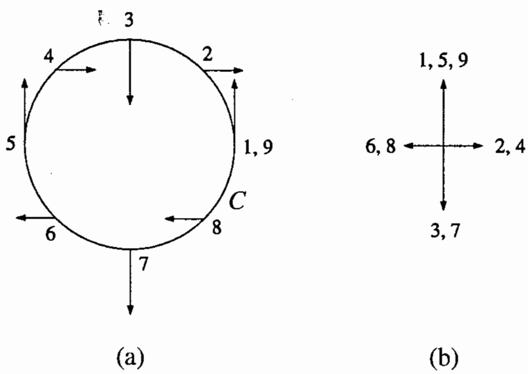

Figure 6.8.5

Now we translate the vectors over to Figure 6.8.5b. As we move from #1 to #9 in order, the vectors rotate 180° clockwise between #1 and #3, then swing back 360° counterclockwise between #3 and #7, and finally rotate 180° clockwise again between #7 and #9 as we complete the circuit of C. Thus $[\phi]_C = -\pi + 2\pi - \pi = 0$ and therefore $I_C = 0$. ■

We plotted nine vectors in this example, but you may want to plot more to see the variation of the vector field in finer detail.

Properties of the Index

Now we list some of the most important properties of the index.

1. Suppose that C can be continuously deformed into C' without passing through a fixed point. Then $I_C = I_{C'}$.

   This property has an elegant proof: Our assumptions imply that as we deform C into C', the index I_C varies *continuously*. But I_C is an integer—hence it can't change without jumping! (To put it more formally, if an integer-valued function is continuous, it must be *constant*.)

   As you think about this argument, try to see where we used the assumption that the intermediate curves don't pass through any fixed points.
2. If C doesn't enclose any fixed points, then $I_C = 0$.

   Proof: By property (1), we can shrink C to a tiny circle without changing the index. But ϕ is essentially constant on such a circle, because all the vectors point in nearly the same direction, thanks to the assumed smoothness of the vector field (Figure 6.8.6). Hence $[\phi]_C = 0$ and therefore $I_C = 0$.

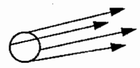

Figure 6.8.6

3. If we reverse all the arrows in the vector field by changing $t \to -t$, the index is unchanged.

   Proof: All angles change from ϕ to $\phi + \pi$. Hence $[\phi]_C$ stays the same.
4. Suppose that the closed curve C is actually a *trajectory* for the system, i.e., C is a closed orbit. Then $I_C = +1$.

   We won't prove this, but it should be clear from geometric intuition (Figure 6.8.7).

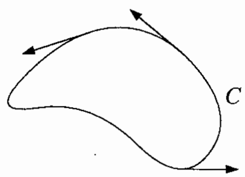

Figure 6.8.7

Notice that the vector field is everywhere tangent to C, because C is a trajectory. Hence, as $\mathbf{x}$ winds around C once, the tangent vector also rotates once in the same sense.

Index of a Point

The properties above are useful in several ways. Perhaps most importantly, they allow us to define the index of a fixed point, as follows.

Suppose $\mathbf{x}^*$ is an isolated fixed point. Then the ***index*** I of $\mathbf{x}^*$ is defined as I_C, where C is *any* closed curve that encloses $\mathbf{x}^*$ and no other fixed points. By property (1) above, I_C is independent of C and is therefore a property of $\mathbf{x}^*$ alone. Therefore we may drop the subscript C and use the notation I for the index of a point.

EXAMPLE 6.8.4:

Find the index of a stable node, an unstable node, and a saddle point.

Solution: The vector field near a stable node looks like the vector field of Example 6.8.1. Hence $I = +1$. The index is also $+1$ for an unstable node, because the only difference is that all the arrows are reversed; by property (3), this doesn't change the index! (This observation shows that *the index is not related to stability,* per se.) Finally, $I=-1$ for a saddle point, because the vector field resembles that discussed in Example 6.8.2. ■

In Exercise 6.8.1, you are asked to show that spirals, centers, degenerate nodes and stars all have $I=+1$. Thus, a saddle point is truly a different animal from all the other familiar types of isolated fixed points.

The index of a curve is related in a beautifully simple way to the indices of the fixed points inside it. This is the content of the following theorem.

Theorem 6.8.1: If a closed curve C surrounds n isolated fixed points $\mathbf{x}_1{}^*, \ldots, \mathbf{x}_n{}^*$, then

$$I_C = I_1 + I_2 + \ldots + I_n$$

where I_k is the index of $\mathbf{x}_k{}^*$, for $k=1, \ldots, n$.

Ideas behind the proof: The argument is a familiar one, and comes up in multivariable calculus, complex variables, electrostatics, and various other subjects. We think of C as a balloon and suck most of the air out it, being careful not to hit any of the fixed points. The result of this deformation is a new closed curve Γ, consisting of n small circles $\gamma_1, \ldots, \gamma_n$ about the fixed points, and two-way bridges connecting these circles (Figure 6.8.8). Note that $I_\Gamma = I_C$, by property (1), since we didn't cross any fixed points during the deformation. Now let's compute I_Γ by considering $[\phi]_\Gamma$. There are contributions to $[\phi]_\Gamma$ from the small circles and from the two-way bridges. The key point is that *the contributions from the bridges cancel out*: as we move around Γ, each bridge is traversed once in one direction, and later in the opposite direction. Thus we only need to consider the contributions from the small circles. On γ_k, the angle ϕ changes by $[\phi]_{\gamma_k} = 2\pi I_k$, by definition of I_k. Hence

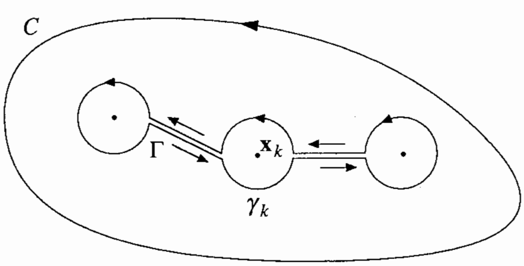

Figure 6.8.8

$$I_\Gamma = \tfrac{1}{2\pi}[\phi]_\Gamma = \tfrac{1}{2\pi}\sum_{k=1}^{n}[\phi]_{\gamma_k} = \sum_{k=1}^{n} I_k$$

and since $I_\Gamma = I_C$, we're done. ■

This theorem is reminiscent of Gauss's law in electrostatics, namely that the electric flux through a surface is proportional to the total charge enclosed. See Exercise 6.8.12 for a further exploration of this analogy between index and charge.

Theorem 6.8.2: Any closed orbit in the phase plane must enclose fixed points whose indices sum to $+1$.

Proof: Let C denote the closed orbit. From property (4) above, $I_C = +1$.

Then Theorem 6.8.1 implies $\sum_{k=1}^{n} I_k = +1$. ■

Theorem 6.8.2 has many practical consequences. For instance, it implies that there is always at least one fixed point inside any closed orbit in the phase plane (as you may have noticed on your own). If there is *only* one fixed point inside, it cannot be a saddle point. Furthermore, Theorem 6.8.2 can sometimes be used to rule out the possible occurrence of closed trajectories, as seen in the following examples.

EXAMPLE 6.8.5:

Show that closed orbits are impossible for the "rabbit vs. sheep" system

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

studied in Section 6.4. Here $x, y \geq 0$.

Solution: As shown previously, the system has four fixed points: $(0,0)$ = unstable node; $(0,2)$ and $(3,0)$ = stable nodes; and $(1,1)$ = saddle point. The index at each of these points is shown in Figure 6.8.9. Now suppose that the system had a closed trajectory. Where could it lie? There are three qualitatively different locations, indicated by the dotted curves C_1, C_2, C_3. They can be ruled out as follows: orbits like C_1 are impossible because they don't enclose any fixed points, and orbits like C_2 violate the requirement that the indices inside must sum to $+1$. But what is wrong with orbits like C_3, which satisfy the index requirement? The trouble is that such orbits always cross the x-axis or the y-axis, and these axes contain straight-line trajectories. Hence C_3 violates the rule that trajectories can't cross (recall Section 6.2). ■

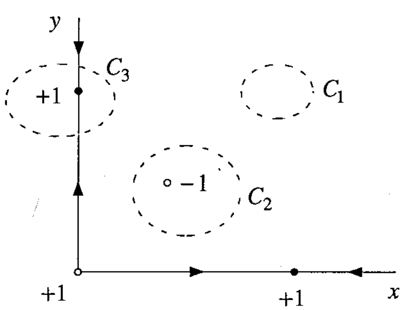

Figure 6.8.9

EXAMPLE 6.8.6:

Show that the system $\dot{x} = xe^{-x}$, $\dot{y} = 1 + x + y^2$ has no closed orbits.

Solution: This system has no fixed points: if $\dot{x} = 0$, then $x = 0$ and so $\dot{y} = 1 + y^2 \neq 0$. By Theorem 6.8.2, closed orbits cannot exist. ■

EXERCISES FOR CHAPTER 6

6.1 Phase Portraits

For each of the following systems, find the fixed points. Then sketch the nullclines, the vector field, and a plausible phase portrait.

6.1.1 $\dot{x} = x - y$, $\dot{y} = 1 - e^x$

6.1.2 $\dot{x} = x - x^3$, $\dot{y} = -y$

6.1.3 $\dot{x} = x(x - y)$, $\dot{y} = y(2x - y)$

6.1.4 $\dot{x} = y$, $\dot{y} = x(1 + y) - 1$

6.1.5 $\dot{x} = x(2 - x - y)$, $\dot{y} = x - y$

6.1.6 $\dot{x} = x^2 - y$, $\dot{y} = x - y$

6.1.7 (Nullcline vs. stable manifold) There's a confusing aspect of Example 6.1.1. The nullcline $\dot{x} = 0$ in Figure 6.1.3 has a similar shape and location as the stable manifold of the saddle, shown in Figure 6.1.4. But they're not the same curve! To clarify the relation between the two curves, sketch both of them on the same phase portrait.

(Computer work) Plot computer-generated phase portraits of the following systems. As always, you may write your own computer programs or use any ready-made software, e.g., *MacMath* (Hubbard and West 1992).

6.1.8 (van der Pol oscillator) $\dot{x} = y$, $\dot{y} = -x + y(1 - x^2)$

6.1.9 (Dipole fixed point) $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$

6.1.10 (Two-eyed monster) $\dot{x} = y + y^2$, $\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borrelli and Coleman 1987, p. 385.)

6.1.11 (Parrot) $\dot{x} = y + y^2$, $\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borrelli and Coleman 1987, p. 384.)

6.1.12 (Saddle connections) A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which

a) there is a single trajectory that connects the saddles;

b) there is no trajectory that connects the saddles.

6.1.13 Draw a phase portrait that has exactly three closed orbits and one fixed point.

6.1.14 (Series approximation for the stable manifold of a saddle point) Recall the system $\dot{x} = x + e^{-y}$, $\dot{y} = -y$ from Example 6.1.1. We showed that this system