MATH 3F03 Assignment 4
Marking Comments

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Preamble

How to use this document: Check your marked assignment solutions for boxed numbers. Each boxed number corresponds to a comment in this document. If any points are deducted from your solution related to a comment, the number of points deducted is includedc in brackets after that comment.

For example,

7. *Page 555, exercise 6.*

> [1] Did not check stability of equilibria **(-2)**

means that in question 7 (which is exercise 6 on page 555 of *Hirsch, Smale, Devaney*), you forgot to check the stability of equilibria, and so I deducted two points.

All questions are marked out of 10.

Note: Only the most commonly applicable comments are included here. Your assignment may have some additional, specific comments.

Comments

1. *Page 157, exercise 2.*

[1] You did not prove your formula for the k^{th} Picard iterate by induction. **(-2)**

[2] Note that X_0 is a vector (written as a column), so for a term like tAX_0 to make sense, X_0 has to be multiplying tA on the right (because of how we define matrix multiplication as rows times columns). As a result, this term written as tX_0A or X_0tA is technically incorrect – instead, if you would like to write X_0 on the left of A, you would have to write $(X_0)^TtA$ or $t(X_0)^TA$ since the transpose of a column would make it a row, and so matrix multiplication would then be well-defined.

[3] It's not true that the $k+1^{\text{th}}$ Picard iterate, $U_{k+1}(t)$, is equal to $\sum_{i=0}^{\infty}\frac{(tA)^i}{i!}X_0$. It's actually that $U_{k+1}(t) = \sum_{i=0}^{k+1}\frac{(tA)^i}{i!}X_0$ (which you can, and should, show by induction) and taking

$\lim_{k\to\infty} U_{k+1}(t)$ gives the solution $X(t)$ to $X' = AX$, which is $\sum_{i=0}^{\infty} \frac{(tA)^i}{i!} X_0 = \exp(tA)X_0$. **(-1)**

4 You need to show that you get the solution $X(t)$ to $X' = AX$ by taking $\lim_{k\to\infty} U_{k+1}(t)$, where $U_{k+1}(t)$ is the $k+1^{\text{th}}$ Picard iterate. **(−1)**.

2. *Page 157, exercise 3.*

1 You're trying to derive the Taylor series of $\sin(2t)$ here, so you cannot appeal to the fact that you know what the Taylor series of $\sin(x)$ looks like. After constructing vector with two series (one in each component) by Picard iteration, you need to show that:

1. The initial value problem

$$X' = AX, \quad X(0) = X_0`$$
$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

   which is a first-order system representing the second order problem

$$x'' = -4x, \quad x(0) = 0, \quad x'(0) = 2,$$

   satisfies the existence and uniqueness theorem, so any solution you find is the only solution.

2. The vector

$$X(t) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

   solves $X' = AX$, and so, by uniqueness of solutions from point 1, it is the only solution.

Only then, when you've shown points 1 and 2, can you conclude that the Taylor series for $\sin(2t)$ is the series in the first component of the vector of two series you found by Picard iteration. **(-2)**

2 You've shown that

$$\tilde{X}(t) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} \tag{1}$$

solves the initial value problem, and you have found series expressions for both components of the vector $X(t)$ through Picard's method, but you haven't shown that the Existence and Uniqueness Theorem applies here. You need the theorem to conclude that, since the initial value problem has a unique solution, then the two solutions you found (*i.e.*,(1) and the vector of series from Picard's method) must be equal. Then you can conclude that the Taylor series for $\sin(2t)$ must be the series in the first component of the vector from Picard's method. **(-1)**

3 You don't actually have to solve the system $X' = AX$ using the Jordan Canonical Form method (and go through somewhat long process of finding eigenvectors) – you just need to show that

$$X(t) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

solves the system (see solutions).

5. *Page 184, exercise 2.*
Generally well done.

6. *Page 184, exercise 12.*
No overall comments because there were many varied answers.

7. *Page 184, exercise 1, parts (a) and (d).*

1 Equilibria only occur when one $x-$ and one $y-$nullcline intersect. If, for instance, two $x-$nullclines intersect, that is not an equilibrium as only one equation (the one for x') is zero. **(-0.5)**

2 Vertical/horizontal directions of vectors don't change except for over nullclines, so you can infer the general behaviour in regions bounded by nullclines based on what happens on the nullclines. You shouldn't have to plug in points and compute the direction of vectors directly.

For instance, if you have one $x-$nullcline, say $y = 0$ (the $x-$axis), and one $y-$nullcline, say $x = 0$ (the $y-$axis), the phase portrait is split up into four major regions (the four quadrants). You can infer the behaviour in each of these four regions based on the behaviour on the nullclines bordering each region.

Suppose we want to figure out what's going on in quadrant I. The bordering nullclines would be the positive $x-$ and $y-$axes. If $x' > 0$ on the positive $y-$axis and $y' < 0$ on the positive $x-$axis, then the overall direction in quadrant I would be southeast. Here's a sketch that fits the above example: