MATH 3F03

Assignment 4 Solutions

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1. pg 157, ex 2

Let A be $n\times n$. Want to show that the Picard method for solving $\mathbf{X}' = A\mathbf{X}$, $\mathbf{X}(0)=\mathbf{X}_0$ gives the solution $\exp(tA)\mathbf{X}_0 = \mathbf{X}(t)$.

Write $\mathbf{X}' = F(\mathbf{X}) = A\mathbf{X}$ (✱). Then the $k+1^{th}$ Picard iterate for this system is given by:

$$\mathbf{U}_{k+1}(t) = \mathbf{X}_0 + \int_0^t F(\mathbf{U}_k(s))\,ds,$$

where $\mathbf{U}_k : \mathbb{R} \to \mathbb{R}^n$ so $F(\mathbf{U}_k(s))$ is a vector of length n & integration is component-wise.

takes in t — gives out n coordinates

recall that $F: \mathbb{R}^n \to \mathbb{R}^n$.

Since $F(\mathbf{X}) = A\mathbf{X}$,

$$\mathbf{U}_{k+1}(t) = \mathbf{X}_0 + \int_0^t A\mathbf{U}_k(s)\,ds.$$

First off, we have

$$\mathbf{U}_0(t) = \mathbf{X}_0.$$

Then $\mathbf{U}_1(t) = \mathbf{X}_0 + \int_0^t A\mathbf{U}_0(s)\,ds = \mathbf{X}_0 + \int_0^t A\mathbf{X}_0\,ds$ (constant with respect to variable of integration s.)

$$= \mathbf{X}_0 + tA\mathbf{X}_0.$$

Next, $\mathbf{U}_2(t) = \mathbf{X}_0 + \int_0^t A\mathbf{U}_1(s)\,ds = \mathbf{X}_0 + \int_0^t A(\mathbf{X}_0 + sA\mathbf{X}_0)\,ds.$

$$= \mathbf{X}_0 + tA\mathbf{X}_0 + \frac{t^2}{2}A^2\mathbf{X}_0$$

Inductively, assume that

$$\mathbf{U}_k(t) = \mathbf{X}_0 + tA\mathbf{X}_0 + \dots + \frac{t^k}{k!}A^k\mathbf{X}_0 \quad (✱)$$

then $\mathbf{U}_{k+1}(t) = \mathbf{X}_0 + \int_0^t A\mathbf{U}_k(s)\,ds = \mathbf{X}_0 + A\int_0^t \mathbf{X}_0 + sA\mathbf{X}_0 + \dots + \frac{s^k}{k!}A^k\mathbf{X}_0\,ds.$

$$= \mathbf{X}_0 + tA\mathbf{X}_0 + \frac{t^2}{2}A^2\mathbf{X}_0 + \dots + \frac{t^{k+1}}{(k+1)!}A^{k+1}\mathbf{X}_0.$$

which satisfies the formula (✱)

Thus, we conclude, by induction, that, the k^{th} term in this sequence of functions $\{\mathbf{U}_k\}$ is $\mathbf{U}_k(t) = \mathbf{X}_0 + tA\mathbf{X}_0 + \dots + \frac{t^k}{k!}A^k\mathbf{X}_0$.

Then, $\mathbf{U}_k(t) = \left(I + tA + \dots + \frac{t^k}{k!}A^k\right)\mathbf{X}_0$

$$= \left(\sum_{i=0}^{k} \frac{(tA)^i}{i!}\right)\mathbf{X}_0$$

Taking $k\to\infty$, $\mathbf{U}_k(t)$ converges to $\left(\sum_{i=0}^{\infty}\frac{(tA)^i}{i!}\right)\mathbf{X}_0 = \exp(tA)\mathbf{X}_0 = \mathbf{X}(t)$ which solves (✱) ■

2. pg. 157, ex 3

(*) $x'' = -4x, \quad x(0) = 0, \quad x'(0) = 2$

(see problem 1)

Note that (*) is a second-order differential equation, but Picard's method only applies to first-order (systems of) equations.
So, we let $y = x'$. Then $y' = x'' = -4x = f(x)$, and we get the first-order system:

$$\mathbf{X}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Thus, using the Picard formula for first-order systems of equations from problem 1, we have that

$$\Pi_{k+1}(t) = \mathbf{X}_0 + \int_0^t A\Pi_k(s)\,ds.$$

Setting $\Pi_0(t) = \mathbf{X}_0$, we get

$$\Pi_1(t) = \mathbf{X}_0 + \int_0^t A\Pi_0(s)\,ds = \mathbf{X}_0 + \int_0^t A\mathbf{X}_0\,ds = \mathbf{X}_0 + tA\mathbf{X}_0$$

$$\Pi_2(t) = \mathbf{X}_0 + \int_0^t A\Pi_1(s)\,ds = \mathbf{X}_0 + \int_0^t A(\mathbf{X}_0 + sA\mathbf{X}_0)\,ds = \mathbf{X}_0 + tA\mathbf{X} + \frac{t^2}{2}A^2\mathbf{X}_0.$$

By the same inductive argument as in problem 2, we get that

$$\Pi_k(t) = \mathbf{X}_0 + tA\mathbf{X}_0 + \frac{t^2}{2}A^2\mathbf{X}_0 + \dots \frac{t^k}{k!}A^k\mathbf{X}_0 = \sum_{i=0}^{k}\frac{(tA)^i}{i!}\mathbf{X}_0$$

& taking $k \to \infty$, we get

$$\mathbf{X}(t) = \left(\sum_{i=0}^{\infty}\frac{(tA)^i}{i!}\right)\mathbf{X}_0 = \mathbf{X}_0 + tA\mathbf{X}_0 + \frac{t^2}{2!}A^2\mathbf{X}_0 + \dots + \frac{t^k}{k!}A^k\mathbf{X}_0 + \dots \quad (*)$$

Taking a closer look at the terms of this series,

$i=1$: $\underbrace{tA\mathbf{X}_0}_{} = t\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = t\underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{} = \begin{pmatrix} 2t \\ 0 \end{pmatrix}$ (†)

$i=2$: $\frac{t^2}{2}A^2\mathbf{X}_0 = \frac{t^2}{2}\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{t^2}{2}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{t^2}{2}(-4)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\Rightarrow A^2 = (-4)I}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$= (-4)\frac{t^2}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{t^2}{2}\cdot 2(-4) \end{pmatrix}$

$i=3$: $\frac{t^3}{3!}A^3\mathbf{X}_0 = \frac{t^3}{3!}\boxed{A^2}\,\underbrace{A\mathbf{X}_0}_{} = \frac{t^3}{3!}\boxed{(-4)I}\underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\text{from } (\dagger)} = \frac{t^3}{3!}(-4)\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{t^3}{3!}(-4)\cdot 2 \\ 0 \end{pmatrix}$

$i=4$: $\frac{t^4}{4!}A^4\mathbf{X}_0 = \frac{t^4}{4!}A^2A^2\mathbf{X}_0 = \frac{t^4}{4!}(-4)I(-4)I\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{t^4}{4!}(-4)^2\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{t^4}{4!}(-4)^2\cdot 2 \end{pmatrix}$

* HINT: when you're trying to find a pattern to derive a series, it's often easier if you don't cancel out constants (even if it's tempting!!)

Putting these terms back into $(*)$, we get

$$\bar{X}(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{t^2}{2}\cdot(-4)\cdot 2 \end{pmatrix} + \begin{pmatrix} \frac{t^3}{3!}\cdot(-4)\cdot 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{t^4}{4!}(-4)^2\cdot 2 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 2t + \frac{t^3}{3!}(-4)\cdot 2 + \cdots \\ 2 + \frac{t^2}{2}(-4)\cdot 2 + \frac{t^4}{4!}(-4)^2\cdot 2 \end{pmatrix} \begin{matrix} \to \text{only odd terms} \\ \to \text{only even terms.} \end{matrix}$$

$$= \begin{pmatrix} 2t + (-1)\frac{t^3}{3!}(2^3) + \cdots \\ 2\left(1 + (-1)\frac{t^2}{2!}\cdot 2^2 + \frac{t^4}{4!}2^4 + \cdots\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i+1}}{(2i+1)!} \\ 2\left(\sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i}}{(2i)!}\right) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} \quad (\heartsuit)$$

Thus, $x(t) = \sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i+1}}{(2i+1)!}$ & $x'(t) = 2\left(\sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i}}{(2i)!}\right)$

Now note that $\bar{X}' = A\bar{X}$ is solved by $\tilde{\bar{X}}(t) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}$, since

$$\tilde{\bar{X}}'(t) = \begin{pmatrix} 2\cos(2t) \\ -4\sin(2t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} = A\tilde{\bar{X}}(t).$$

& $\tilde{\bar{X}}(0) = \begin{pmatrix} \sin(2\cdot 0) \\ 2\cos(2\cdot 0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, so $\tilde{\bar{X}}(t)$ satisfies the initial value problem $\bar{X}' = A\bar{X}$, $\bar{X}(0) = \bar{X}_0$.

Lastly, we note that $F(\bar{X}) = A\bar{X}$ is C^1 since A induces a linear transformation on each component of $\bar{X}$ (& so it is, in fact, C^∞), & so by the existence & uniqueness theorem $\tilde{\bar{X}}(t)$ is the <u>only</u> solution to this IVP. Thus, it must be that $\tilde{\bar{X}}(t)$ is equal to the series derived above using the Picard method. $(\heartsuit)$

In other words, $\tilde{\bar{X}}(t) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i+1}}{(2i+1)!} \\ 2\left(\sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i}}{(2i)!}\right) \end{pmatrix} = \bar{X}(t).$

Therefore, $\boxed{\sin(2t) = \sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i+1}}{(2i+1)!}}$

& $\cancel{2}\cos(2t) = \cancel{2}\left(\sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i}}{(2i)!}\right) \Rightarrow \boxed{\cos(2t) = \sum_{i=0}^{\infty} \frac{(-1)^i (2t)^{2i}}{(2i)!}}$

3. pg 157, ex 6

Consider $x' = x^a$ where $a > 0$. If Case ①: $a \geq 1$, $f_a(x) = x^a$ is C^1, & so by the existence & uniqueness theorem, the initial value problem $x' = f_a(x)$, $x(0) = x_0$ has a unique solution in some time interval $(-\varepsilon, \varepsilon)$.
In particular, we would have a unique solution for the IVP with $x(0) = 0$.
Case ②: If $0 < a < 1$, $f_a(x) = x^a$ is no longer differentiable at $x = 0$, (it is not C^1) & so the existence & uniqueness theorem applies.
However, for some $0 < a < 1$ (like for $a = \frac{1}{2}$), $f_a(x)$ is not even defined for $x < 0$, but we want to consider $x(0) = 0$ ... to avoid this problem, let's instead consider an extended version of $f_a(x) = x^a \rightarrow \tilde{f}_a(x) = |x|^a$.
Then, $\tilde{f}_a = |x|^a$ is continuous for all $x \in \mathbb{R}$. Thus, by the Global Picard theorem, there exists a solution $y : (-\delta, \delta) \rightarrow \mathbb{R}$ for some neighbourhood $(-\delta, \delta)$ of the initial time $t_0 = 0$. For instance, $x \equiv 0$ (i.e. $x(t) = 0$ for all time) satisfies $x' = \tilde{f}_a(x) = |x|^a$ for $x(0) = 0$, for all time (i.e. $t \in (-\delta, \delta) = (-\infty, \infty) = \mathbb{R}$).

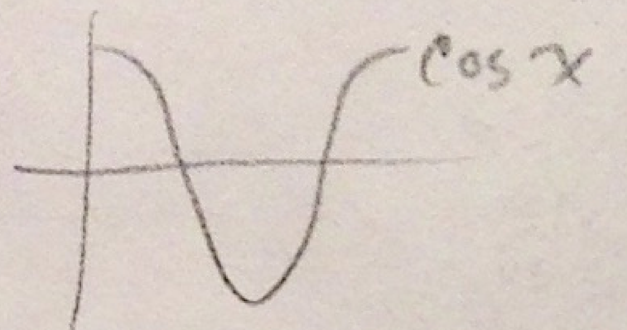
4. pg 184, ex 1 (a & c)

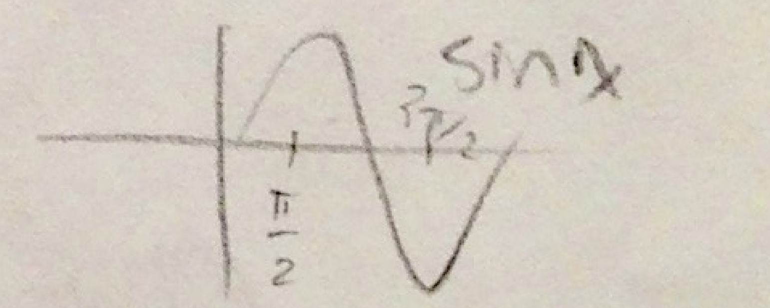
i. $x' = \sin x = f(x)$, $y' = \cos(y) = g(y)$

a. $x' = \sin x = 0$ when $x = n\pi$ for $n \in \mathbb{Z}$; $y' = \cos(y) = 0$ when $y = (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$ $\Big\}$ equilibrium points are of the form $\underline{X}^*_n = (x,y)^*_n = \left(n\pi, (2n+1)\frac{\pi}{2}\right)$ for $n \in \mathbb{Z}$

The associated linear system about the equilibrium $\underline{X}^*$ is given by $\underline{X}' = DF_{\underline{X}^*}\underline{X}$ where $DF_{\underline{X}^*}$ is the Jacobian evaluated at $\underline{X}^*$.

$$DF_{\underline{X}} = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix}, \text{ in general.}$$





Here,
$$DF_{\underline{X}^*_n} = \begin{pmatrix} \cos(x) & 0 \\ 0 & -\sin(y) \end{pmatrix}\Bigg|_{\underline{X}^*_n} = \begin{pmatrix} \cos(n\pi) & 0 \\ 0 & -\sin\left((2n+1)\frac{\pi}{2}\right) \end{pmatrix}$$

$$= \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n \text{ is even.} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } n \text{ is odd.} \end{cases}$$

If n is even $\Rightarrow \lambda_1 = 1$ so $alg(\lambda_1) = 2$ but $geom(\lambda_1) = 2$ since $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ special source

If n is odd $\Rightarrow \mu_1 = -1$ so $alg(\mu_1) = 2$ but $geom(\mu_1) = 2$ since $W_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $W_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\Rightarrow$ special sink.

c. Since both $Re(\lambda_1)$ & $Re(\mu_1) \neq 0$ for any n, the equilibrium $\bar{X}_n^*$ is hyperbolic for all n & so the linearized system accurately describes the local behaviour of the non-linear system near the equilibrium points.

v. $x' = x^2 = f(x)$, $y' = y^2 = g(y)$

a. $\left.\begin{array}{l} x' = x^2 = 0 \iff x = 0 \\ y' = y^2 = 0 \iff y = 0 \end{array}\right\}$ $(x,y) = (0,0)$ is the only equilibrium point.

$DF_{\bar{X}} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$, so $DF_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda = 0.$

The linearized system $\bar{X}' = DF_{(0,0)}\bar{X}$ is the trivial system (every point is an equilibrium).

c. The linear system does not accurately describe the local behaviour of the non-linear system near the equilibrium $(0,0)$, since $(0,0)$ is not a hyperbolic equilibrium ($Re(\lambda) = 0$).

(You can find an online direction field generator to plot the direction field of $x' = x^2$, $y' = y^2$ to see that, near $(0,0)$, the system is definitely not trivial).

5. pg 184, ex 2

Want a global change of variables that linearizes the system

$$(*) \begin{cases} x' = x + y^2 \\ y' = -y \\ z' = -z + y^2 \end{cases}$$

Note that in section 8.1, we see that the two-dimensional sub-system $\begin{cases} x' = x + y^2 \\ y' = -y \end{cases}$ has a global change of variables that linearizes the system.

This change of variables is $\begin{bmatrix} u = x + \frac{1}{3}y^2 \\ v = -y \end{bmatrix}$

Note that we can use these variables to linearize part of $(*)$, but we must find one more variable, $w = f(y,z)$ to linearize the last equation in $(*)$.

This equation, $z' = -z + y^2$, is of a similar form as $x' = x + y^2$, the latter being linearized by $u = x + \frac{1}{3}y^2$ & $v = -y$, so it would make sense to choose a w similar to u.

Let's try $w = z + ay^2$ for some $a > 0$. (**Note**: I did have to play around with this to find a would have to be positive!)

Then $w' = z' + 2ayy'$

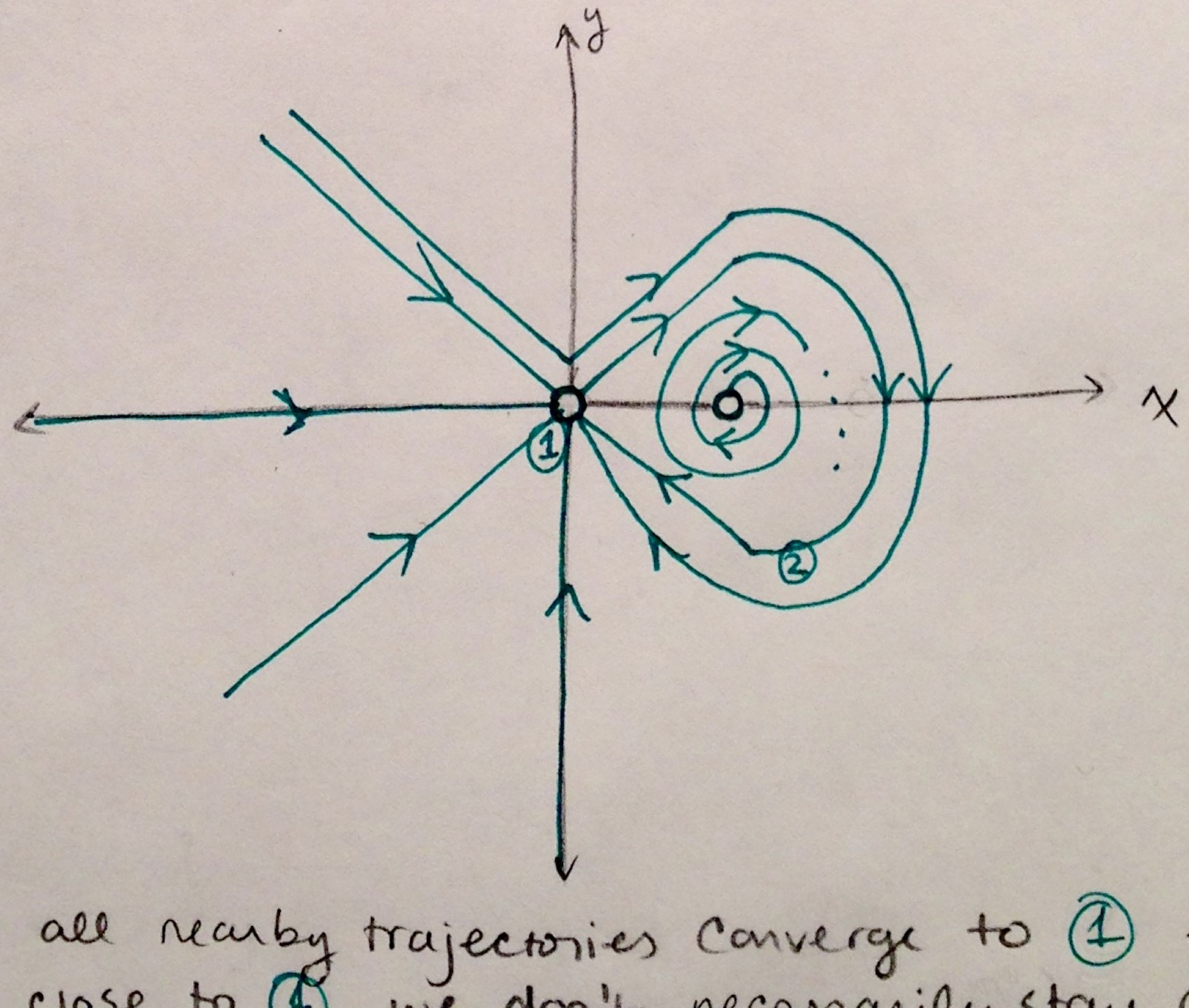
$$= -z + y^2 - 2ay^2 = -(z + [2a-1]y^2) \quad (*)$$

For the right-hand side of (*) to relate back to w, we need $2a-1=1$, which would imply that $w' = -w$. But that's possible if $a=1$, & so we have our change of variables. Namely, letting $\begin{bmatrix} u = x + \frac{1}{3}y^2 \\ v = -y \\ w = z + y^2 \end{bmatrix}$, the non-linear system $\begin{Bmatrix} x' = x + y^2 \\ y' = -y \\ z' = z - y^2 \end{Bmatrix}$

is transformed into the linear system $\begin{Bmatrix} u' = u \\ v' = -v \\ w' = -w \end{Bmatrix}$

6. pg. 186, ex 12

We want to sketch a system with an equilibrium point x^* that has the property that all nearby solution curves (eventually) tend to x^*, but x^* is not stable by the precise definition of "stable" to show that convergence to an equilibrium point does NOT imply that this equilibrium is stable. The following sketch shows such an $\underbrace{\text{equilibrium point}}_{\text{labelled } ①}$:



So, although all nearby trajectories converge to ① eventually, if we start close to ①, we don't necessarily stay close for all $t>0$. (in particular if we start in quadrant ①).
If you still don't see why ① is not stable, look at the definition of stability & draw an open ball about ① that intersects trajectory ② (which is a special type of trajectory called a homoclinic orbit)

7. pg. 210, ex 1.

a. $x' = x(y+2x-2) = 0$ when $\underbrace{x=0}_{X1}$, OR when $y+2x-2=0$

$\underbrace{y = 2-2x}_{X2}$

x-nullclines

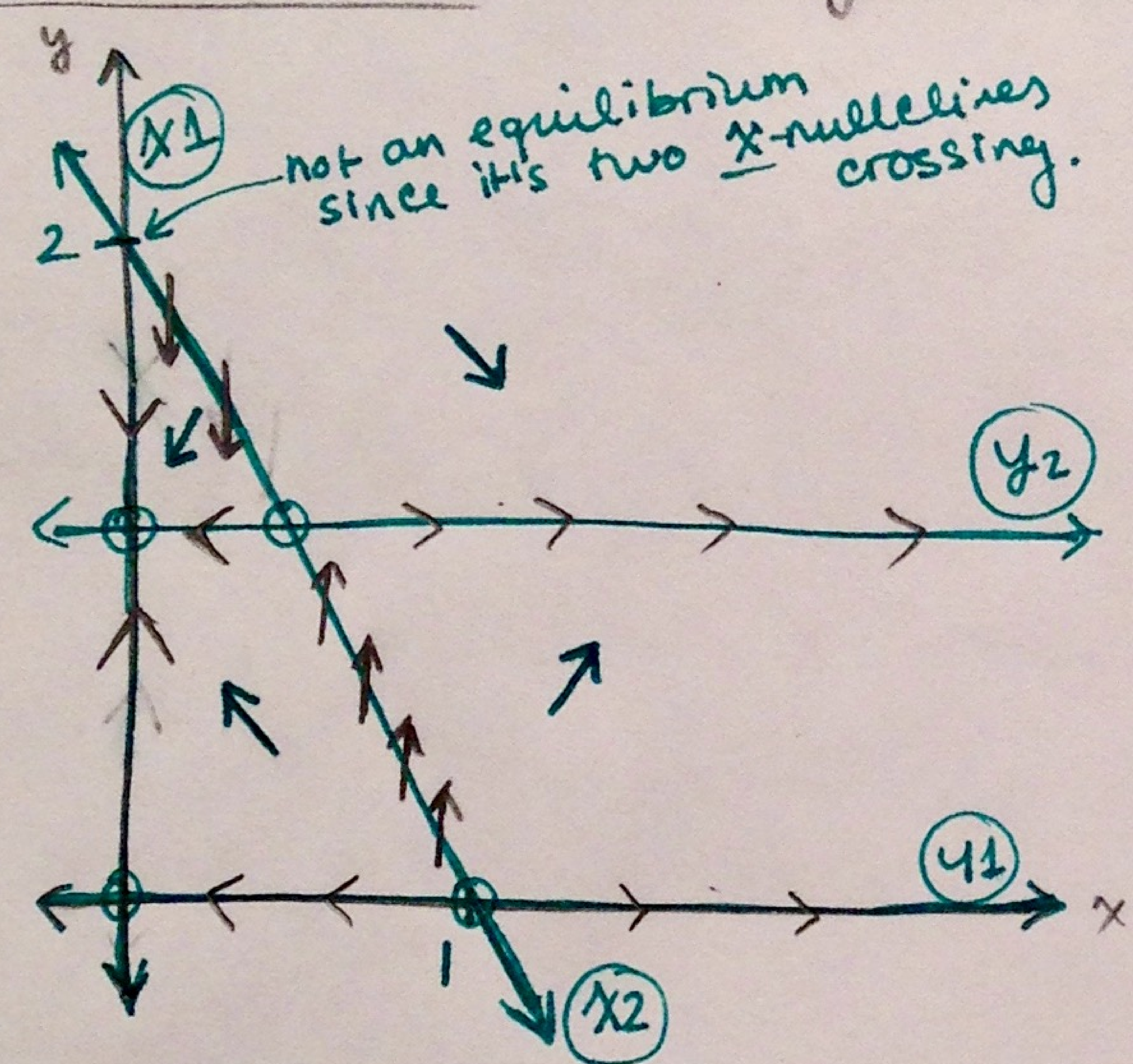
$y' = y(y-1) = 0$ when $\underbrace{y=0}_{Y1}$, OR when $y-1=0$

$\underbrace{y=1}_{Y2}$

y-nullclines

* Equilibria occur when x- & y-nullclines intersect.

Phase Portrait (for $x, y \geq 0$, as the exercise suggests)



- for x-nullclines, only vertical movement.
  - on X1: $y' > 0$ if $y > 1$
    $y' < 0$ if $0 < y < 1$
  - on X2: same as for X1
- for y-nullclines, only horizontal movement.
  - on Y1: $y = 0$, so $x' > 0$ if $x > 1$
    $x' < 0$ if $0 < x < 1$
  - on Y2: $y = 1$, so $x' > 0$ if $x > \frac{1}{2}$
    $x' < 0$ if $0 < x < \frac{1}{2}$

d. $x' = x(2-y-2x) = 0$ when $\underbrace{x=0}_{X1}$, OR when $2-y-2x=0$

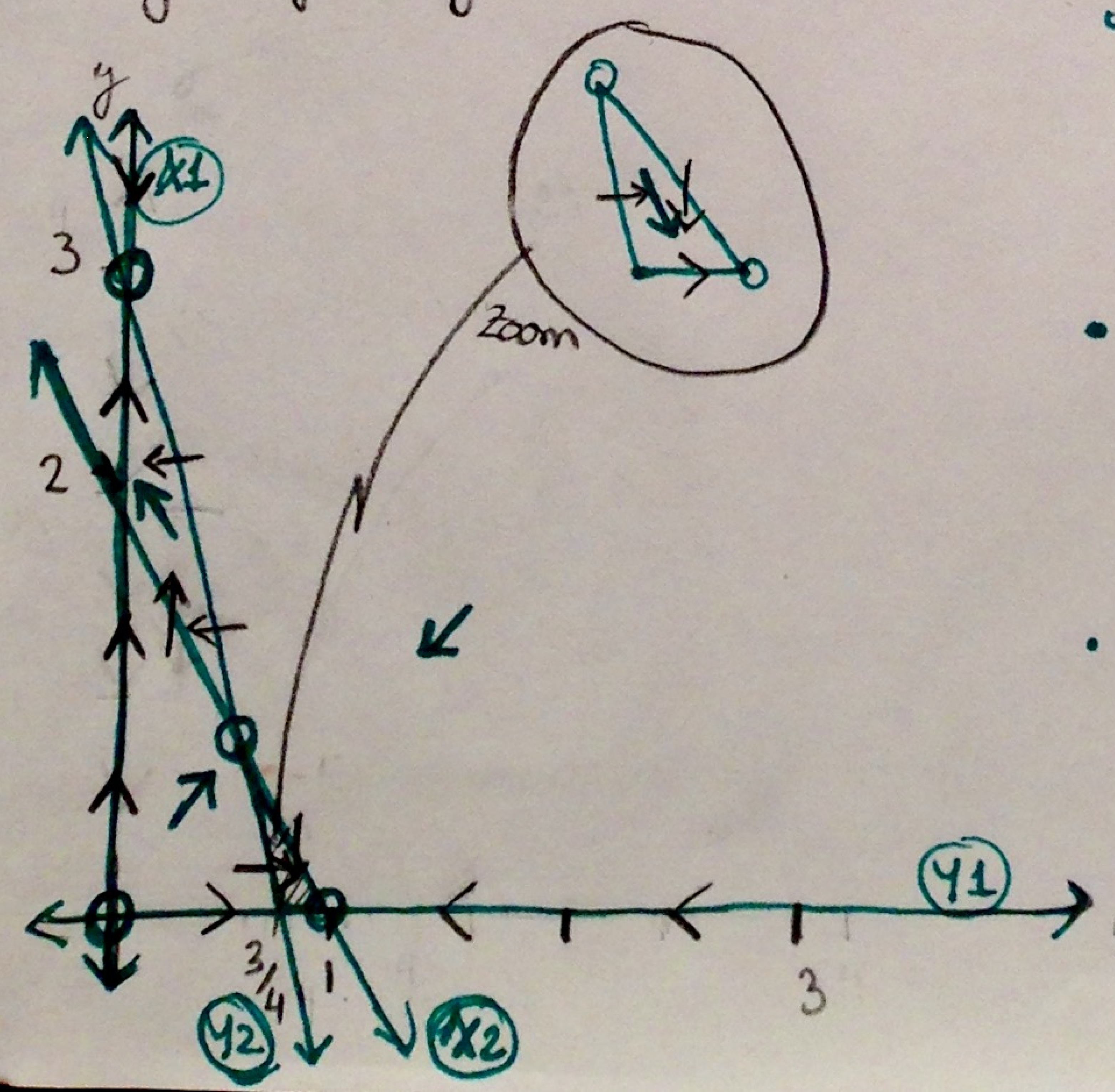
$\underbrace{y = 2-2x}_{X2}$

x-nullclines

$y' = y(3-y-4x) = 0$ when $\underbrace{y=0}_{Y1}$, OR when $3-y-4x=0$

$\underbrace{y = 3-4x}_{Y2}$

y-nullclines



- for x-nullclines:
  - on X1: $x = 0$, so $y' < 0$ if $y > 3$, $y' > 0$ if $0 < y < 3$
  - on X2: $y' > 0$ if $y < 3-4x$, $y' < 0$ if $y > 3-4x$.
- for y-nullclines:
  - on Y1: $y = 0$, so $x' > 0$ if $0 < x < 1$
    $x' < 0$ if $x > 1$
  - on Y2: $x' > 0$ if $y < 2-2x$
    $x' < 0$ if $y > 2-2x$.