Solutions

1.) $\det(A-\lambda I) = \lambda^2+12\lambda+72 \Longrightarrow \lambda = -6 \pm 6i$

JCF of $A = J = \begin{pmatrix} -6 & 6 \\ -6 & -6 \end{pmatrix}$

Eigenvector of $\lambda=-6+6i$ is $\begin{pmatrix} 2 \\ 3i \end{pmatrix}$, so a T such that

$T^{-1}AT = J$ is $T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, so $T^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

Then we know $e^J = e^{T^{-1}AT} = T^{-1}e^A T \Longrightarrow$

$$e^A = Te^J T^{-1}$$

To compute e^J, write $J = U+V$, where

$U = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix}$

Note U and V commute since U is a multiple of the identity matrix

Then from class we know $e^J = e^{U+V} = e^U e^V$

$e^U = \begin{pmatrix} e^{-6} & 0 \\ 0 & e^{-6} \end{pmatrix}$ (U is diagonal)

$e^V = \begin{pmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{pmatrix}$ (we did this in class)

So $e^J = \begin{pmatrix} e^{-6} & 0 \\ 0 & e^{-6} \end{pmatrix}\begin{pmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{pmatrix} = e^{-6}\begin{pmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{pmatrix}$

Then $e^A = Te^J T^{-1}$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\left(e^{-6}\begin{pmatrix} \cos 6 & \sin 6 \\ -\sin 6 & \cos 6 \end{pmatrix}\right)\left(\tfrac{1}{6}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right)$$

$$= e^{-6}\begin{pmatrix} \cos 6 & \tfrac{2}{3}\sin 6 \\ -\tfrac{3}{2}\sin 6 & \cos 6 \end{pmatrix}$$

2.) (i) $u_0 = 2$

$u_1 = 2 + \int_0^t (2+2)\,ds = 2 + 4t$

$u_2 = 2 + \int_0^t (2+2+4s)\,ds = 2 + 4t + 2t^2$

$u_3 = 2 + \int_0^t (2 + 2 + 4s + 2s^2)\,ds = 2 + 4t + 2t^2 + \frac{2}{3}t^3$

(ii) $u_k(t) = 2 + \left(4 \sum_{j=1}^{k} \frac{t^j}{j!}\right) = \left(4 \sum_{j=1}^{k} \frac{t^j}{j!}\right) - 2$

(iii) $\lim_{k \to \infty} u_k(t) = 4e^t - 2$

3.) (i) A unique solution is guaranteed by the Fundamental Existence and Uniqueness Theorem only when $|x|^{3/2+\sin a}$ is C^1 on some neighborhood of $(0,\vec{0})$.

For $|x|^{3/2+\sin a}$ to be C^1, we must have

$$\frac{3}{2} + \sin a > 1 \iff \sin a > -\frac{1}{2}$$

$$\iff \boxed{a \in [0, 5\pi/6) \cup (7\pi/6, 2\pi)}$$

(ii) For the remaining values of a, $a \in [\frac{5\pi}{6}, \frac{7\pi}{6}]$, we have $\frac{3}{2} + \sin a \le 1 \implies |x|^{3/2+\sin a}$ is continuous, but not differentiable.

Hence for $a \in [5\pi/6, 7\pi/6]$ the Peano Existence Theorem gives us a solution which is not necessarily unique.

4.) (i) Equilibria where $x' = 0 = y'$

$x' = 0 \Rightarrow \cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, \; n \in \mathbb{Z}$

$y' = 0 \Rightarrow \sin y = 0 \Rightarrow y = m\pi, \; m \in \mathbb{Z}$

Hence all equilibria have the form

$$\left((2n+1)\frac{\pi}{2}, \; m\pi\right), \quad n, m \in \mathbb{Z}$$

(ii) The linearized system at an equilibrium (x^*, y^*) is $X' = AX$, where

$$A = DF\Big|_{(x^*, y^*)} = \left.\begin{pmatrix} \frac{\partial}{\partial x}(\cos x) & \frac{\partial}{\partial y}(\cos x) \\ \frac{\partial}{\partial x}(\sin y) & \frac{\partial}{\partial y}(\sin y) \end{pmatrix}\right|_{(x^*, y^*)}$$

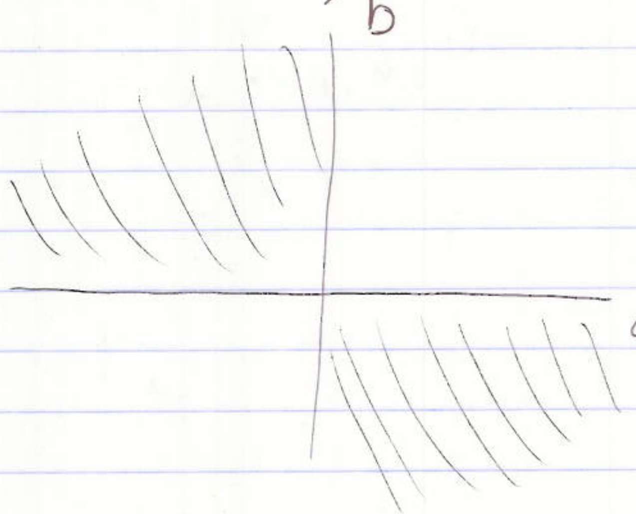
$$= \left.\begin{pmatrix} -\sin x & 0 \\ 0 & \cos y \end{pmatrix}\right|_{(x^*, y^*)}$$

$$= \begin{pmatrix} -\sin((2n+1)\frac{\pi}{2}) & 0 \\ 0 & \cos(m\pi) \end{pmatrix}, \; n, m \in \mathbb{Z}$$

$$= \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & n, m \text{ even} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & n \text{ odd}, m \text{ even} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & n, m \text{ odd} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & n \text{ even}, m \text{ odd} \end{cases}$$

(iii) Yes, because all equilibria are hyperbolic

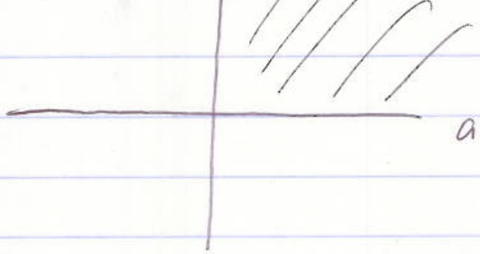
5.) (i) spiral saddle where $a, b \neq 0$, and
either $a < 0$, $b > 0$
or $a > 0$, $b < 0$



(Quadrants 2 and 4 in ab-plane, excluding axes)

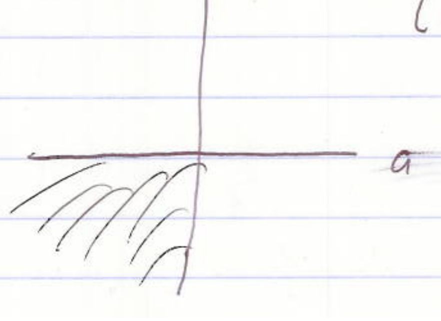
(ii) We have a spiral center where $a = 0$, $b \neq 0$

(iii) source where $a, b > 0$



(Quadrant 1 in ab-plane)

(iv) sink where $a, b < 0$



(Quadrant 3 in ab-plane)