December 4, 2019

ArtsSci 1D06 • Review Problems for Midyear Exam

1. Find the domains of $f(x) = \sqrt{\ln x - 5}$ and $g(x) = \frac{2x^2+3x-5}{e^{2x}+e^x-6}$ and determine all their x-intercepts.

2. (a) Evaluate $\sin(\tan^{-1}(\frac{40}{41}))$.
(b) Find all solutions to $\sin(2x) = \tan(x)$ with $-\pi/2 < x < \pi/2$.

3. Suppose $f(x)$ is even and $g(x), h(x)$ are both odd. In (a)–(d), determine whether the given function is even, odd, or neither.
(a) $f(x) - g(x)$
(b) $g(x) + 2h(x)$
(c) $f(x)g(x)$
(d) $g(x)/h(x)$

4. In (a)–(f), evaluate the following limit if it exists. If the limit is infinite, work out whether the answer is $+\infty$ or $-\infty$. Otherwise, if the limit does not exist, write DNE.

(a) $\lim_{x\to 2}\left(\frac{x^2-4}{x^3-8}\right)$

(b) $\lim_{x\to 2}\left(\frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x}\right)$

(c) $\lim_{x\to\infty}\tan^{-1}(x-x^3)$

(d) $\lim_{x\to 0^+}\left(\frac{x}{\ln x}\right)$

(e) $\lim_{x\to 0}(1-3x)^{1/x}$

(f) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

(g) $\lim_{x\to\infty}\left(x\ln(1+\frac{3}{x})\right)$

(h) $\lim_{x\to 0}\left(\frac{\sinh 2x}{x^3-3x^2+5x}\right)$

5. In (a)–(f), find $\frac{dy}{dx}$. Simplify your answer if possible.
(a) $y = \tan(1-x^2)$
(b) $y = \sqrt{4+\sin x\cos x}$
(c) $y = \ln(\ln(\ln x))$
(d) $y = \tan^{-1}(\sinh x)$
(e) $y = x^{\sqrt{x}}$
(f) $\sec(xy) = x^2 - y$
(g) $y = \dfrac{x+1}{\tanh(x)}$
(h) $y = \sin(x^3) + \sin^3(x)$

6. Given $f(x)$ and $g(x)$ differentiable functions with $f(1) = 5,\ g(1) = 2,\ f'(1) = a$, and $g'(1) = b$, find the slope of the line tangent to $y = f(x)g(x^2) + f(x^2)g(x)$ at $(1, 20)$ in terms of the constants a, b.

7. Find all critical values of the function $f(x) = \frac{4}{3}x^3 + 2x^2 - 3x + 2$ and classify them as local maximum or local minimum. Find all inflection points.

8. Construct the linear approximation to $f(x) = \sqrt[5]{1+2x}$ at $a = 0$ and use it to approximate the value $\sqrt[5]{1.02}$.

9. Given a function $f(x)$ with $f'(2) = \sqrt{5}$, evaluate the limit $\lim_{x\to 2}\dfrac{f(x)-f(2)}{x^3-8}$.

10. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

11. Show that $f(x) = x^3 - 7x^2 + 25x + 84$ has exactly one real root.

12. For the following functions, find all local extrema, inflection points, intervals of increase/decrease, intervals of concave up/down, vertical and horizontal asymptotes, and x- and y-intercepts. Using this informtion, sketch the curve $y = f(x)$.

(a) $f(x) = x^5 - 5x$

(b) $f(x) = 2 - 2x - x^3$

(c) $f(x) = \dfrac{x}{1-x^2}$

(d) $f(x) = \dfrac{x^3-1}{x^3+1}$

(e) $f(x) = x\sqrt{2+x}$

(f) $f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

13. Use Newton's method with $x_0 = 1$ to find a solution to $x \cos x = x^2$ correct to three decimal places.

14. Show that the equation $x^{99} + 2x^{55} + 3x - 5 = 0$ has exactly one real root.

15. Find (a) $\dfrac{d}{dx}\displaystyle\int_0^{\ln x} t^2 + 1\, dt.$ (b) $\displaystyle\int_1^3 \left(\frac{d}{dx}\sqrt{\ln x}\right) dx.$

16. In (a)–(f), evaluate the indefinite integral.

(a) $\displaystyle\int (x+2)^{19}\, dx$

(b) $\displaystyle\int xe^{x^2+1}\, dx$

(c) $\displaystyle\int \frac{x^2}{x^3+1}\, dx$

(d) $\displaystyle\int \sin \pi t \cos \pi t\, dt$

(e) $\displaystyle\int \frac{\sec t \tan t}{1+\sec t}\, dt$

(f) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\, dx$

17. (a) Show that $\ln x - \ln a = \displaystyle\int_a^x \frac{1}{t}\, dt$ for all $x > a > 0$.

(b) Find $\displaystyle\lim_{x\to a} \frac{1}{x-a}\int_a^x \frac{1}{t}\, dt$ for $a > 0$.

(c) Evaluate $\displaystyle\lim_{x\to a} \frac{1}{x-a}\int_a^x f(t)\, dt$ if $f(t)$ is defined and continuous for all real numbers.

18. Determine the limit

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^3} + \frac{2^3}{n^3} + \frac{3^3}{n^3} + \cdots + \frac{n^3}{n^3}\right].$$

19. (a) Find the area bounded by the curves $y = 1 + \sqrt{x}$ and $y = (3+x)/3$.

(b) Find the area bounded by the curves $x + y = 0$ and $x = y^2 + 3y$.

20. A particle moves along a line with velocity $v(t) = t^3 - 9t$, where v is measured in meters per second. Find the displacement and distance traveled by the particle during the time interval $[1, 4]$.

21. At which points on the curve $y = 1 + 30x^3 - x^5$ does the tangent line have largest slope?

22. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

23. Prove the identity $\sin^{-1}\left(\dfrac{x-1}{x+1}\right) = 2\tan^{-1}(\sqrt{x}) - \dfrac{\pi}{2}$.