Quiz 7a **ArtsSci 1D06** NAME: ______________

February 7, 2020 Student Number: ______________

1. Evaluate the integral $\int (\ln(x))^3\, dx$.

$u = (\ln(x))^3 \quad dv = dx$

$du = \frac{3(\ln x)^2}{x} dx \quad v = x$

$*\quad u = (\ln(x))^2 \quad dv = dx$

$du = \frac{2\ln(x)}{x} dx \quad v = x$

$$\int \ln(x)^3 dx = x(\ln(x))^3 - \int 3(\ln(x))^2 dx \; *$$

$$= x(\ln(x))^3 - 3\left[x(\ln(x))^2 - \int 2\ln x\, dx\right]$$

$$= x(\ln(x))^3 - 3x(\ln(x))^2 + 6x\ln x - 6x + C$$

2. Evaluate the integral $\int_0^1 \sqrt{x - x^2}\, dx$.

Complete the square for $x - x^2 = \frac{1}{4} - \frac{1}{4} + x - x^2$

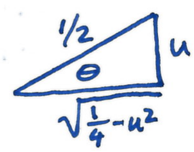
$= \frac{1}{4} - (x - \frac{1}{2})^2$

$$\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{\tfrac{1}{4} - (x-\tfrac{1}{2})^2}\, dx$$

$u = x - \frac{1}{2}$

$du = dx$

$$= \int_{-1/2}^{1/2} \sqrt{\tfrac{1}{4} - u^2}\, du$$



$\sin\theta = \frac{u}{1/2} = 2u$

$\cos\theta = \frac{\sqrt{1/4 - u^2}}{1/2} = 2\sqrt{1/4 - u^2}$

$du = \frac{1}{2}\cos\theta\, d\theta$

$u = \pm 1/2 \Rightarrow \theta = \sin^{-1}(\pm 1)$

$= \pm\frac{\pi}{2}$

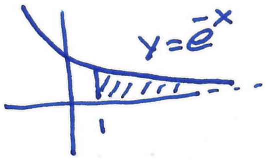
$$= \int_{-\pi/2}^{\pi/2} \tfrac{1}{4}\cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{8}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{8}\left[\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0\right)\right]$$

$$= \frac{\pi}{8}$$

3. Find the area in the region bounded by the x-axis, the curve $y = e^{-x}$ for $x \geq 1$.



$$\text{Area} = \int_1^\infty e^{-x}\,dx \text{. This is an improper int'l,}$$

$$= \lim_{t\to\infty} \int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_1^t$$

$$= \lim_{t\to\infty} \left[-e^{-t} + e^{-1}\right] = e^{-1}$$

since $\lim_{t\to\infty} e^{-t} = 0$

4. Evaulate the integral $\int \frac{dx}{(1+\sqrt{x})^2}$.

$u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow dx = 2u\,du$

$$\int \frac{2u\,du}{(1+u)^2}$$

We use partial fractions...

$$\frac{2u}{(1+u)^2} = \frac{A}{1+u} + \frac{B}{(1+u)^2}$$

$2u = A(1+u) + B \Rightarrow A = 2 ; B = -2$

$$\int \frac{2u\,du}{(1+u)^2} = \int \frac{2\,du}{1+u} - \int \frac{2\,du}{(1+u)^2}$$

$$= 2\ln|1+u| + 2(1+u)^{-1} + C$$

$$= 2\ln|1+\sqrt{x}| + \frac{2}{(1+\sqrt{x})} + C$$

Quiz 7b ArtsSci 1D06 NAME: ______________________

February 7, 2020 Student Number: ______________________

1. Evaluate the integral $\int \sqrt{x}e^{\sqrt{x}}\,dx$.

$u=\sqrt{x},\ u^2=x \Rightarrow dx = 2u\,du$

$\int u e^{u} \cdot 2u\,du = 2\int u^2 e^u\,du$

$w=u^2 \quad dw = 2u\,du$

$dv = e^u du \quad v = e^u$

$= 2u^2e^u - 4\int ue^u\,du$

$= 2u^2e^u - 4ue^u + 4u + C$

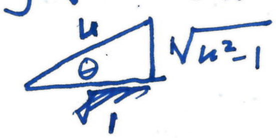
$= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4\sqrt{x} + C$

2. Evaluate the integral $\int \sqrt{x^2+2x}\,dx$.

Complete the square: $x^2+2x = (x+1)^2 - 1$

$\int \sqrt{(x+1)^2-1}\,dx = \int \sqrt{u^2-1}\,du \qquad u = x+1,\ du = dx$

Use trig. subst.



$\sec\theta = u$

$\tan\theta = \sqrt{u^2-1}$

$du = \sec\theta\tan\theta\,d\theta$

$\int \tan\theta \cdot \sec\theta\tan\theta\,d\theta$

$= \int \sec\theta\tan^2\theta\,d\theta \qquad \tan^2\theta = \sec^2\theta - 1$

$= \int (\sec^3\theta - \sec\theta)\,d\theta$

$= \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta| + C$

(see p. 483)

$= \frac{1}{2}(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|) + C$

$= \frac{1}{2}(u\sqrt{u^2-1} - \ln|u+\sqrt{u^2-1}| + C)$

$= \frac{1}{2}((x+1)\sqrt{x^2+2x} - \ln|x+1+\sqrt{x^2+2x}|) + C$

3. Does the improper integral $\int_1^{\infty} \frac{dx}{\sqrt{1+x^4}}$ converge? Why or why not?

Using comparison with $f(x) = \frac{1}{x^2}$, we see that for $x \geq 1$, $\frac{1}{\sqrt{1+x^4}} \leq \frac{1}{x^2}$. Therefore,

$$\int_1^{\infty} \frac{dx}{\sqrt{1+x^4}} \leq \int_1^{\infty} \frac{dx}{x^2}$$ which converges since

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^2} = \lim_{t\to\infty} \left[-x^{-1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} + 1\right] = 1$$

4. Evaluate the integral $\int \frac{1-\tan^2(x)}{\sec^2(x)}\, dx$.

$$= \int \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}\, dx = \int (\cos^2 x - \sin^2 x)\, dx$$

$$= \int (1 - 2\sin^2 x)\, dx = x - 2\int \sin^2 x\, dx$$

$$= x - 2\int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= x - x + \frac{1}{2}\sin 2x + C$$

$$= \sin x \cos x + C$$

Quiz 7c ArtsSci 1D06 NAME: ____________________

February 7, 2020 Student Number: ____________________

1. Evaluate the integral $\int x^3 e^x \, dx$.

$u = x^3 \quad dv = e^x \, dx$

$du = 3x^2 dx \quad v = e^x$

$\int x^3 e^x dx = x^3 e^x - 3\int x^2 e^x dx \qquad u = x^2 \quad du = 2x\,dx \quad dv = e^x dx \quad v = e^x$

$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$

2. Evaluate the integral $\int \frac{x^2+1}{x^2-3x} \, dx$.

$\frac{x^2+1}{x^2-3x} = \frac{x^2-3x+3x+1}{x^2-3x} = 1 + \frac{3x+1}{x(x-3)}$

$\frac{3x+1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} \Rightarrow 3x+1 = A(x-3) + Bx$

$x=0 \Rightarrow 1 = -3A \Rightarrow A = \frac{1}{3}$

$x=3 \Rightarrow 10 = 3B \Rightarrow B = \frac{10}{3}$

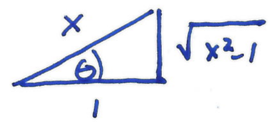
$\int \frac{x^2+1}{x^2-3x} dx = \int 1 \, dx + \frac{1}{3}\int \frac{dx}{x} + \frac{10}{3}\int \frac{dx}{x-3}$

$= x + \frac{1}{3}\ln|x| + \frac{10}{3}\ln|x-3| + C$

3. Find the average value of the function $f(x) = \dfrac{\sqrt{x^2-1}}{x}$ over the interval $1 \le x \le 7$.

$$\text{avg value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{6}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx$$



$\tan\theta = \sqrt{x^2-1}$

$\sec\theta = x$

$dx = \sec\theta\tan\theta\,d\theta$

$$= \frac{1}{6}\int_{x=1}^{x=7} \frac{\tan\theta \cdot \sec\theta\tan\theta}{\sec\theta}\,d\theta = \frac{1}{6}\int_{x=1}^{x=7} \tan^2\theta\,d\theta$$

$$= \frac{1}{6}\int_{x=1}^{x=7} (\sec^2\theta - 1)\,d\theta$$

$$= \frac{1}{6}\left[\theta + \tan\theta\right]$$

$$= \frac{1}{6}\left[\cos^{-1}\left(\frac{1}{x}\right) + \sqrt{x^2-1}\right]_1^7$$

$$= \frac{1}{6}\left[\sqrt{48} - \cos^{-1}\left(\frac{1}{7}\right)\right]$$

4. Does the improper integral $\displaystyle\int_1^\infty \frac{x}{1+x^3}\,dx$ converge? Why or why not?

Use comparison with $\int_1^\infty \frac{1}{x^2}dx$ to show it converges.

$$\frac{x}{1+x^3} \le \frac{x}{x^3} = \frac{1}{x^2} \quad \text{for } 1 \le x,$$

Also, $\displaystyle\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$

$$= \lim_{t\to\infty}\left[-\frac{1}{t} + 1\right] = 1 \quad \text{converges.}$$

Therefore, by the comparison theorem, $\displaystyle\int_1^\infty \frac{x}{1+x^3}\,dx$ also converges.