11.10 Taylor and Maclaurin Series

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series which converges for $|x-a|<R$.

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Then

$$f(a) = c_0$$
$$f'(x) = c_1 + c_2 \cdot 2(x-a) + c_3 \cdot 3 \cdot (x-a)^2 + \cdots$$
$$f'(a) = c_1$$
$$f''(x) = 2c_2 + 2 \cdot 3 \cdot c_3 (x-a) + \cdots$$
$$f''(a) = 2c_2$$
$$f^{(3)}(x) = 2 \cdot 3 \cdot c_3 + \cdots$$
$$f^{(3)}(a) = 2 \cdot 3 \cdot c_3$$
$$\vdots$$
$$f^{(n)}(a) = 2 \cdot 3 \cdots n \cdot c_n = n!\, c_n$$

Hence $c_n = \dfrac{f^{(n)}(a)}{n!}$ for $n = 0, 1, 2, \ldots$

and

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}$$

This formula is the <u>Taylor series</u> of $f(x)$ centered at a.

The special case $a = 0$ is called the <u>Maclaurin series</u>, and it is given by

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

<u>Example</u> $f(x) = e^x$

$$f'(x) = e^x = f''(x) = f^{(3)}(x) = \dots = f^{(n)}(x)$$

Therefore $f^{(n)}(0) = e^0 = 1$ for all $n = 0, 1, 2, \dots$

So e^x has <u>Maclaurin series</u> $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

It's Taylor series at a is given by

$$\sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$

<u>Question</u>: If $f(x)$ is infinitely differentiable, that is if $f^{(n)}(x)$ exists for all $n = 0, 1, 2, \dots$ when is its Taylor series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n$

a <u>convergent</u> power series (i.e. one with $R > 0$)?

Does $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

<u>Example</u> (counterexample)

$$\text{Consider} \quad f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function is infinitely differentiable at $x = 0$ (this is not obvious but it is true), and in fact $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \dots$

(Exercise: prove that $f^{(n)}(0) = 0$ using L'Hopital's Rule.)

Therefore its Maclaurin series is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$ for all x. But $f(x)$ is not identically zero, i.e. for all $x_0 \neq 0$, $f(x_0) = e^{-1/x_0^2} \neq 0$.

<u>Example</u> $f(x) = \sin(x) \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(iv)}(x) = \sin x \qquad f^{iv}(0) = 0$

$f(x)$ has Maclaurin series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

This series will be shown to have radius of convergence $R = \infty$.

<u>Example</u> $g(x) = \cos(x) \qquad g(0) = 1$

$g'(x) = -\sin x \qquad g'(0) = 0$

$g''(x) = -\cos x \qquad g''(0) = -1$

$g'''(x) = \sin x \qquad g'''(0) = 0$

$g^{iv}(x) = \cos x \qquad g^{iv}(0) = 1$

$g(x)$ has Maclaurin series $\displaystyle\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!}$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Alternatively, one can derive this from the power series expansion for $f(x) = \sin x$ by differentiating term by term.

$$\boxed{\begin{aligned} \sin x &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}}$$

11.10 Taylor and Maclaurin Series (cont'd)

Recall if $f(x)$ is infinitely differentiable, its Taylor series centered at a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Question: When does the Taylor series converge to $f(x)$?

Defn The n-th partial sum of the Taylor series is called the n-th order Taylor polynomial of $f(x)$.

$$T_n(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

In particular $T_0(x) = f(a)$ — constant

$T_1(x) = f(a) + f'(a)(x-a)$ — eqn of tangent line at a

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

We would like to know whether $\{T_n(x)\}_{n=1}^{\infty}$ provide a good approximation to $f(x)$, i.e.

does $\lim_{n\to\infty} T_n(x) = f(x)$?

The remainder $R_n(x) = f(x) - T_n(x)$ measures how well $T_n(x)$ approximates $f(x)$.

If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$ (where $R>0$ or $R=\infty$)

then $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}(f(x) - R_n(x)) = f(x) - \lim_{n\to\infty} R_n(x)$

$= f(x)$.

In other words, $f(x)$ is equal to its Taylor series for $|x-a|<R$

In order to use $T_n(x)$ to approximate $f(x)$, we need an estimate for the error term $R_n(x)$.

<u>Taylor's Inequality</u> If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.

<u>Proof</u> (sketch, $n=1$ case only)

Assume $|f''(x)| \leq M$, so $-M \leq f''(x) \leq M$.

Then $\underbrace{\int_a^x f''(t)\,dt}_{\text{By FTC } f'(x)-f'(a)} \leq \int_a^x M\,dt = M(x-a)$ for $0 \leq x-a \leq d$

So

$$f'(x) \leq f'(a) + M(x-a)$$

Apply $\int_a^x \ldots dt$ again...

$$\underbrace{\int_a^x f'(t)\,dt}_{} \leq \int_a^x [f'(a) + M(t-a)]\,dt$$

$$f(x) - f(a) \leq f'(a)(x-a) + \frac{M}{2}(x-a)^2$$

$$f(x) \leq \underbrace{f(a) + f'(a)(x-a)}_{T_1(x)} + \frac{M}{2}(x-a)^2$$

$$R_1(x) = f(x) - T_1(x) \leq \frac{M}{2}(x-a)^2$$

In a similar way, can show $R_1(x) \geq -\frac{M}{2}(x-a)^2$

Thus

$$|R_1(x)| \leq \frac{M}{2}|x-a|^2$$

Applications of Taylor inequality

1. $f(x) = e^x$

$f^{(n+1)}(x) = e^x$ so $\max\{|f^{(n+1)}(x)| \mid |x| \le d\} = e^d$

(since e^x is increasing)

By Taylor's inequality $|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$

For any x, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ so $\lim_{n\to\infty} R_n(x) = 0$

<u>Conclusion</u> e^x is equal to its Maclaurin series.

2. $f(x) = \sin(x)$

Then $f^{(n+1)}(x) = \begin{cases} \pm \sin x & \text{if } n \text{ odd} \\ \pm \cos x & \text{if } n \text{ even} \end{cases}$

In either case, $|f^{(n+1)}(x)| \le 1$ for all x.

By Taylor's inequality $|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$ for all x.

Hence $\lim_{n\to\infty} R_n(x) = 0$ so $\sin(x)$ is also equal to its Maclaurin series. (i.e. Radius of conv. $R = \infty$).

3. Similarly, $g(x) = \cos x$ can be seen to be equal to its Maclaurin series

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{(infinite radius of conv)}$$