Chapter 11 - Infinite sequences and series.

11.1 – 11.7 deal with sequences, series, convergence tests

11.8 - 11.11 deal with power series; Taylor and Maclaurin series.

11.1 Sequences

A sequence is an infinite list of real numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Notation $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$

Here a_n is the <u>nth term</u> in the sequence and n is the <u>index</u>.

Examples.

1. $\frac{1}{n}$ Harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$
3. $\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty} = \{1, 3, 6, 10, \ldots\}$
4. $\{\cos n\pi\}_{n=1}^{\infty} = \{-1, +1, -1, +1, \ldots\}$

   since $\cos n\pi = (-1)^n$

Sometimes it is convenient to start a sequence at an integer different from 1.

<u>Ex</u> $\{\sqrt{n-3}\}_{n=3}^{\infty}$ is the sequence $0, 1, \sqrt{2}, \sqrt{3}, \sqrt{4}=2, \ldots$

Sometimes we specify a sequence without giving a formula for the nth term.

For example, the Fibonacci sequence is defined by setting $a_1 = 1 = a_2$ and declaring that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.
This is a recursively defined sequence, given by $1, 1, 2, 3, 5, 8, 13, \ldots$

Defn A sequence $\{a_n\}$ has limit L, written $\lim_{n\to\infty} a_n = L$, if we can make a_n as close to L as we like by choosing n sufficiently large.

If $\lim_{n\to\infty} a_n = L$, we say the sequence converges.

If the limit does not exist, we say the sequence diverges.

Examples

1. $\lim_{n\to\infty} \frac{1}{n} = 0$
2. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0} = 1$
3. $\lim_{n\to\infty} (-1)^n$ does not exist.

Therefore, the first two sequences converge, but the third sequence diverges.

4. $\lim_{n\to\infty} n^2+3 = \infty$

When the limit equals $\pm\infty$, this is a special case.

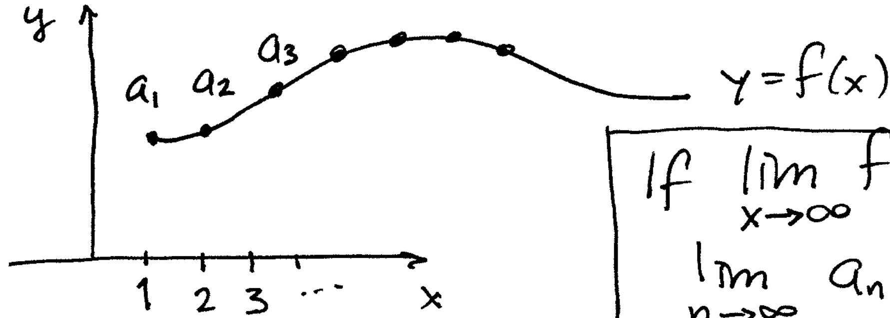
<u>Defn</u> We say $\lim_{n\to\infty} a_n = \infty$ if we can make a_n as large as we like by choosing n sufficiently large. In that case, we say that the sequence diverges to ∞.

Not all sequences diverge in this way. For instance, the sequence $(-1)^n$, which is called the alternating sequence, diverges but does not tend to ∞.

Note. $\lim_{n\to\infty} a_n = -\infty$ is defined similarly.

<u>Example</u> Consider the sequence $\ln(\frac{1}{n})$. Then it is not difficult to see that $\lim_{n\to\infty} \ln(\frac{1}{n}) = -\infty$. Thus this sequence diverges to $-\infty$.

<u>Remark</u> If $f(x)$ is a function defined on $[1,\infty)$, then setting $a_n = f(n)$ for $n = 1, 2, 3, \ldots$ gives a sequence.



$$\boxed{\text{If } \lim_{x\to\infty} f(x) = L \text{ then } \lim_{n\to\infty} a_n = L}$$

Thus evaluating a limit for a sequence is the same as determining if the graph of $y=f(x)$ has a horizontal asymptote.

Limit Rules

If a_n, b_n are convergent, then

(1) a_n+b_n is convergent, with $\lim\limits_{n\to\infty}(a_n+b_n)=\lim\limits_{n\to\infty}a_n+\lim\limits_{n\to\infty}b_n$

(2) a_n-b_n —"—, with $\lim\limits_{n\to\infty}(a_n-b_n)=\lim\limits_{n\to\infty}a_n-\lim\limits_{n\to\infty}b_n$

(3) $a_n\cdot b_n$ —"—, with $\lim\limits_{n\to\infty}(a_n b_n)=\lim\limits_{n\to\infty}a_n \lim\limits_{n\to\infty}b_n$

(4) If $\lim\limits_{n\to\infty}b_n\neq 0$, then $\frac{a_n}{b_n}$ is convergent and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}$$

(5) If c is a constant, then $\lim\limits_{n\to\infty}ca_n=c\lim\limits_{n\to\infty}a_n$

(6) If $f:\mathbb{R}\to\mathbb{R}$ is continuous, then

$$\lim_{n\to\infty}f(a_n)=f\left(\lim_{n\to\infty}a_n\right).$$

Example Let $f(x)=x^2$ and set $a_n=\frac{1}{n}$ and $b_n=\frac{-n}{3n+2}$. Then $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{n}=0$

So by (6) above, $\lim\limits_{n\to\infty}f(a_n)=f(0)=0$

Likewise, we see that $\lim\limits_{n\to\infty}\frac{-n}{3n+2}=\lim\limits_{n\to\infty}\frac{-1}{3+2/n}=\frac{-1}{3+0}=\frac{-1}{3}$

Therefore $\lim\limits_{n\to\infty}f(b_n)=f\left(\lim\limits_{n\to\infty}b_n\right)=f\left(-\frac{1}{3}\right)=\frac{1}{9}$.

Example Let $a_n = \sin(\frac{\pi}{n})$.
Since $\sin(x)$ is continuous for all x,
$\lim_{n\to\infty} \sin(\frac{\pi}{n}) = \sin(\lim_{n\to\infty} \frac{\pi}{n}) = \sin(0) = 0$.

SANDWICH (or SQUEEZE) THEOREM.

If $a_n \le b_n \le c_n$ for all $n \geq N$,
and if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$,

then $\lim_{n\to\infty} b_n = L$.

Example Let $a_n = \frac{n!}{n^n}$, where $n! = 1\cdot 2\cdot 3\cdots n$.
Then $0 \le a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdots n} \le \frac{1}{n}$
since each quotient $\frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}$ is less than or equal to 1. Therefore, by the sandwich theorem, since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$,
it follows that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.