11.1 continued. ⑥

Last time. Introduced sequences $\{a_n\}$, discussed convergence.

Important theorem: If $\lim_{n\to\infty} a_n = L$ and $f: \mathbb{R} \to \mathbb{R}$ is continuous,

then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$.

Sequences in popular math:

Digits of irrational numbers such as π, $\sqrt{2}$, and e.

$\pi = 3.1415629535\ldots$ to 10 decimal places

[side note: math enthusiasts have computed π to 31.4 TRILLION decimal places, according to GOOGLE, March 2019].

One can define a sequence using the digits of π by setting a_n equal to π to n digits.

So
$$a_1 = 3.1$$
$$a_2 = 3.14$$
$$a_3 = 3.141$$
$$a_4 = 3.1415$$

and so on. <u>Question</u>: if $\{a_n\}$ is a sequence defined this way, is it really true that $\lim_{n\to\infty} a_n = \pi$?

<u>Answer</u>: It <u>better</u> be true!

To explain this properly, we need to introduce a few new concepts.

<u>Defn</u> A sequence is <u>monotone</u> <u>increasing</u> if

$$a_1 \leq a_2 \leq a_3 \leq \ldots$$

<u>Defn</u> A sequence is bounded from above if there exists M with $a_n \leq M$ for all n.

Basic Property of Real numbers is that any monotone increasing sequence that is bounded from above is <u>convergent</u>.

<u>Defn</u> A sequence is <u>monotone</u> <u>decreasing</u> if

$$a_1 \geq a_2 \geq a_3 \geq \ldots$$

<u>Defn</u> A sequence is bounded from below if there exists m with $m \leq a_n$ for all n.

Basic property of Real numbers is that any monotone decreasing sequence that is bounded from below is <u>convergent</u>.

<u>Theorem</u> Every bounded monotone sequence is convergent.

Applications

Ex 1 Let $a_1 = 2$ and define a_n recursively by setting

$$a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for } n \geq 2.$$

Using induction, one can prove that $a_n < 6$, so a_n is bounded.

Also using induction, one can show $a_n < a_{n+1}$, so it is monotone.

By theorem $\lim_{n\to\infty} a_n = L$, ie this sequence is convergent.

How to evaluate limit? Use the following trick.

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) \quad \text{by recursion}$$

$$= \frac{1}{2}\lim_{n\to\infty} a_n + 3 \quad \text{by Limit Laws}$$

$$= \frac{1}{2}L + 3$$

Therefore $L = \frac{1}{2}L + 3 \Rightarrow \frac{1}{2}L = 3 \Rightarrow L = 6$.

Ex 2 Define $a_1 = \sqrt{2}$
and $a_{n+1} = \sqrt{2 + a_n}$

One can prove by induction that $1 < a_n < 2$ for all n
so a_n is bounded.
One can also prove that a_n is monotone increasing.
[This step is harder than in Ex 1.]
Therefore by the theorem, $\lim_{n\to\infty} a_n = L$, i.e. sequence converges.
We use the same type of trick to determine L.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} \quad \text{by recursion}$$

$$= \sqrt{2 + \lim_{n\to\infty} a_n} \quad \text{since } f(x) = \sqrt{2+x} \text{ is } \underline{\text{continuous}}.$$

$$= \sqrt{2+L}$$

This gives an equation: $L = \sqrt{2+L}$ or $L^2 = 2+L$
$0 = L^2 - L - 2 = (L-2)(L+1) \Rightarrow L = 2$ or $L = -1$.
We exclude $L = -1$ since we know $L \geq 0$.
Therefore, $\lim_{n\to\infty} a_n = 2$.