11.2 Series

Given a sequence $a_1, a_2, \ldots, a_n, \ldots$ we can form a <u>series</u> by taking the infinit sum $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Σ-notation: (warning!)

We write $\sum_{n=1}^{\infty} a_n$ for the infinite sum $a_1 + a_2 + \cdots + a_n + \cdots$

What exactly does this mean?

This is a question that has puzzled mathematicians and philosophers for many centuries.

[See Zeno's paradoxes, esp. Achilles and the tortoise.]

Q: How can we make sense out of an infinite sum?

Bad news: $\sum_{n=1}^{\infty} a_n$ does not always make sense!

[Niels Hendrik Abel (1826) wrote "Divergent series are the invention of the devil"!]

A: We form a <u>sequence</u> of partial sums $S_n = \sum_{i=1}^{n} a_i$ and take limit of S_n as $n \to \infty$.

[Analogy with improper integrals of the form $\int_1^{\infty} f(x)\,dx$.]

Given infinite series $\sum_{n=1}^{\infty} a_n$

Let $S_n = \sum_{i=1}^{n} a_i$ and define $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i \right)$

provided the limit exists and is finite.

We say $\sum_{n=1}^{\infty} a_n$ <u>converges</u> if S_n converges, and that $\sum_{n=1}^{\infty} a_n$ <u>diverges</u> if S_n diverges.

In particular, if the partial sums S_n converge with $\lim_{n\to\infty} S_n = L$, then we set $\sum_{n=1}^{\infty} a_n = L$.

Examples

1. You have been secretely using infinite series every time you write a real number as a decimal expansion.

For instance, consider $\pi = 3.14159\ 26535\ldots$

Define a sequence $\{d_i\}_{i=0}^{\infty}$ with $d_0 = 3$, $d_1 = 1$, $d_2 = 4$, $d_3 = 1$, etc.

So d_n is the n^{th} digit in the decimal expansion for π.

Thus $\pi = d_0 + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \cdots + \frac{d_n}{(10)^n} + \cdots$ (*)

(Note that each d_i is a number between 0 & 9.)

We can rewrite (*) using Σ-notation as

$$\pi = \sum_{n=0}^{\infty} \frac{d_n}{10^n}$$

Notice that this series starts at $n=0$ (different but okay)

and note the convention/rule $10^0 = 1$.

The partial sum $S_n = \sum_{i=0}^{n} \frac{d_i}{10^i}$ is the value of π up to n-decimal places.

Example Geometric Series

Let $a \neq 0$ and r be numbers, and consider the series

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \quad \left(= \sum_{n=0}^{\infty} ar^n\right)$$

(Convention: $r^0 = 1$)

This is called the geometric series.

Its nth partial sum is $S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + \cdots + ar^{n-1}$

$$rS_n = r\left(\sum_{i=1}^{n} ar^{i-1}\right) = ar + ar^2 + \cdots + ar^n$$

Subtracting, notice the massive cancellation

$$S_n - rS_n = (a + ar + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n)$$

$$= a - ar^n$$

Therefore $(1-r)S_n = S_n - rS_n = a - ar^n = a(1-r^n)$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r} \quad \text{provided } r \neq 1.$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \left(\frac{a}{1-r}\right) \lim_{n\to\infty} (1-r^n)$$

Further: $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$.

and it is undefined if $|r| \geq 1$.

Conclusion: The geometric series converges if $|r| < 1$

with $\boxed{\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}}$

Otherwise if $|r| \geq 1$, it diverges.

<u>Examples</u> 1. $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$ is a geom. series with $a=1$, $r=\frac{1}{3}$

therefore it converges and $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{1-1/3} = \frac{1}{2/3} = 3/2$.

<u>Exercise</u>: Show $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n} = \frac{1}{2}$.

2. $\sum_{n=1}^{\infty} 2^n 3^{1-n} = \sum_{n=1}^{\infty} 2\cdot\left(\frac{2}{3}\right)^{n-1}$ is a geom. series with $a=2$; $r=\frac{2}{3}$

$\Rightarrow$ convergent

$$= 2\left(\frac{1}{1-2/3}\right) = 2\left(\frac{1}{1/3}\right) = 2\cdot 3 = 6.$$

3. Repeating decimals can be converted to fractions using this formula.

Consider $0.17\overline{17} = 0.17171717\ldots$

This is equal to the series $\frac{17}{100} + \frac{17}{10^4} + \frac{17}{10^6} + \ldots = \sum_{n=1}^{\infty}\frac{17}{10^{2n}} = \sum_{n=1}^{\infty}\frac{17}{(100)^n}$

<u>geometric series</u> with $a=\frac{17}{100}$ and $r=\frac{1}{100}$ $\qquad = \sum_{n=1}^{\infty}\frac{17}{100}\cdot\left(\frac{1}{100}\right)^{n-1}$

Therefore it is convergent with limit

$$\frac{17}{100}\cdot\left(\frac{1}{1-1/100}\right) = \frac{17}{100}\cdot\left(\frac{1}{99/100}\right) = \frac{17}{\cancel{100}}\cdot\frac{\cancel{100}}{99} = \frac{17}{99}.$$

In the next example, we apply the method of <u>partial fractions</u> 5

to the telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Using partial fractions, we can write $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$.

Solving for A, B, one gets $A = 1$ and $B = -1$.

Therefore $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \dots$

The partial sum $S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$ (after massive cancellation!)

Clearly $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1$

So the telescoping series converges and we find

that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Divergence Test

Theorem If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

This is intuitively clear, but we will give a short proof momentarily

Application (contrapositive)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Examples. 1. $\sum_{n=1}^{\infty} \frac{n}{n+1}$. Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, this series diverges.

2. $\sum_{n=1}^{\infty} (-1)^n$ Since $\lim_{n\to\infty} (-1)^n$ does not exist, this series diverges.

WARNING It is possible for $\lim_{n\to\infty} a_n = 0$ and the series $\sum_{n=1}^{\infty} a_n$ diverges, i.e. the test above only is conclusive when $\lim_{n\to\infty} a_n \neq 0$.

Example Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

Clearly $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$, so this series passes the test.

However,
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$
$$\geq 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots = 1 + \frac{1}{2} + \frac{1}{2} + \dots$$

Therefore $S_{2^n} = 1 + \frac{n}{2}$ which tends to ∞ as $n \to \infty \Rightarrow$ DIVERGENT!

Limit Rules

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent, then so are $\sum_{n=1}^{\infty}(a_n+b_n)$, $\sum_{n=1}^{\infty}(a_n-b_n)$, $\sum_{n=1}^{\infty} ca_n$. Further, we have

(1) $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(2) $\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

(3) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

Example $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$

$$= \frac{1}{2}\left(\frac{1}{1-1/2}\right) + 1/3\left(\frac{1}{1-1/3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1/2}\right) + 1/3\left(\frac{1}{2/3}\right)$$

$$= \frac{1}{2}\cdot 2 + \frac{1}{3}\cdot\frac{3}{2}$$

$$= 1 + \frac{1}{2} = \frac{3}{2}.$$