Recap: Given a series $\sum_{n=1}^{\infty} a_n$ (*)

We would like to know if (*) converges or diverges.

Ex: Geometric series $\sum_{n=0}^{\infty} r^n = \begin{cases} \text{converges to } \frac{1}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$

In general, if (*) converges, then $\lim_{n\to\infty} a_n = 0$

Div. Test: If $\lim_{n\to\infty} a_n \neq 0$, then (*) diverges.

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## 11.3 Integral Test

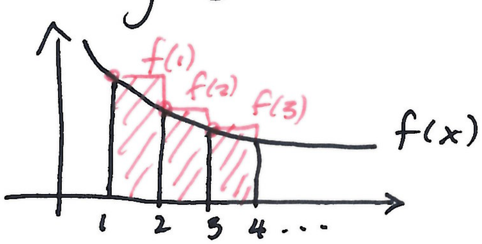
We will use improper integrals $\int_1^{\infty} f(x)\,dx$ to determine when a series $\sum_{n=1}^{\infty} a_n$ converges. (Assumption: $f(n) = a_n$.)

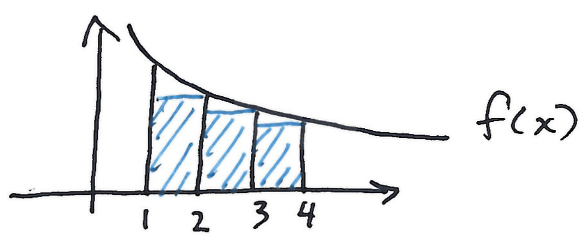
Henceforth suppose $a_n \geq 0$ in (*).

Also suppose $a_n = f(x)$ for function $f(x)$ defined for $x \geq 1$.

**Integral Test** If $f(x)$ is a positive, decreasing function defined for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Why is this true?




$\sum_{n=1}^{N} a_n \geq \int_1^N f(x)\,dx$

$\sum_{n=2}^{N} a_n \leq \int_1^N f(x)\,dx$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is finite if and only if $\int_1^{\infty} f(x)\,dx$ is finite

Examples

1. This test applies to show the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

How? Let $f(x) = \frac{1}{x}$.
The integral test tells us $\sum_{n=1}^{\infty} \frac{1}{n}$ converges if and only if $\int_1^{\infty} \frac{1}{x}\,dx$ converges.

However,
$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\ln|x|\right]_1^t$$
$$= \lim_{t\to\infty} (\ln t - \cancel{\ln 1}^{\,0}) = \infty$$

Since this is not finite, we conclude $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

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2. We can apply the same reasoning to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Let $f(x) = \frac{1}{x^2}$.
Then
$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{x}\right]_1^t$$
$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1$$

Since the improper integral converges, by the integral test we see that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges.

Remark: ① $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$, however $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

② Same argument applies to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $0<p$ and shows it converges iff $p > 1$.

Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Example (cont'd)

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ We use the integral test to show it converges.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$$
$$= \lim_{t\to\infty} \left[\tan^{-1}x\right]_1^t$$
$$= \lim_{t\to\infty} \left[\tan^{-1}t - \tan^{-1}1\right]$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Since this imp. int'l converges, we conclude $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

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4. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Using $f(x) = \frac{1}{x \ln x}$, we evaluate the improper integral

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \left[\ln(\ln x)\right]_2^t$$

$u = \ln x$
$du = \frac{1}{x}dx$
$\int \frac{1}{u}\,du = \ln|u| + C$
$= \ln(\ln x) + C$

$$= \lim_{t\to\infty} \left[\ln(\ln t) - \ln(\ln 2)\right]$$
$$= \infty$$

Therefore, the integral test applies to show $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges.