# 11.4 Comparison Tests

Recap: • $\sum_{n=1}^{\infty} \frac{1}{n^p}$ - p-series converges if $p>1$
diverges if $0<p\leq 1$

• $\sum_{n=1}^{\infty} ar^{n-1}$ geometric series - converges if $|r|<1$
diverges if $|r|>1$.

(*) $\sum_{n=1}^{\infty} a_n$ series with $a_n \geq 0$. Q: Does it converge or diverge?

**Comparison Test** Suppose $\sum a_n$, $\sum b_n$ are two series with
$0 \leq a_n \leq b_n$.
(1) If $\sum b_n$ converges, then $\sum a_n$ converges
(2) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

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**Example** $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. compare to $\sum \frac{1}{n^2}$, p-series with $p=2$ so it converges.

Let $a_n = \frac{1}{n^2+1}$, $b_n = \frac{1}{n^2}$.

Notice $n^2+1 > n^2 \Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}$.

Therefore, since $\sum \frac{1}{n^2}$ converges, the Comp. Thm. implies $\sum \frac{1}{n^2+1}$ also converges.

Remark: we knew this already!

<u>Remark</u>: In applying the comparison theorem, it is enough to know that $0 \le a_n \le b_n$ for all $n \geq N_0$, where $N_0$ is a fixed integer.

That is because $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N_0}^{\infty} a_n$ converges.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, ie harmonic series.

For $n \geq 3$, $\ln n > 1$. $\Rightarrow$ $\frac{\ln n}{n} > \frac{1}{n}$.

Take $a_n = \frac{1}{n}$ and $b_n = \frac{\ln n}{n}$. Then $0 \le \frac{1}{n} \le \frac{\ln n}{n}$

and since $\sum \frac{1}{n}$ diverges, we see that $\sum \frac{\ln n}{n}$ also diverges.

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<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+n+1}}$ compare to $\sum \frac{1}{n^{5/2}}$ p-series with $p = 5/2 > 1$, which converges.

$$n^5+n+1 > n^5$$

$$\Rightarrow \sqrt{n^5+n+1} > n^{5/2} \Rightarrow \frac{1}{\sqrt{n^5+n+1}} < \frac{1}{n^{5/2}}$$

Taking $a_n = \frac{1}{\sqrt{n^5+n+1}}$, $b_n = \frac{1}{n^{5/2}}$ then $0 \le a_n < b_n$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges, this implies that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+n+1}}$ also converges.

Why is the comparison test true?

If $\sum_{n=1}^{\infty} a_n$ is a series with $a_n \geq 0$,

then the partial sums $S_N = \sum_{n=1}^{N} a_n$ forms a monotone increasing seq.

Indeed, $S_{N+1} = \sum_{n=1}^{N+1} a_n = \sum_{n=1}^{N} a_n + a_{N+1} = S_N + a_{N+1} \geq S_N$.

i.e. $S_1 \leq S_2 \leq S_3 \leq \dots \leq S_N \leq S_{N+1} \leq \dots$

So $\sum_{n=1}^{\infty} a_n$ converges if and only if $\{S_N\}_{N=1}^{\infty}$ is <u>bounded</u>.

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Suppose $0 \leq a_n \leq b_n$ $(\dagger)$

Set $S_N = \sum_{n=1}^{N} a_n$ and $S_N' = \sum_{n=1}^{N} b_n$.

then clearly $S_N \leq S_N'$ by condition $(\dagger)$ above.

So $\{S_N'\}_{N=1}^{\infty}$ bounded $\Rightarrow$ $\{S_N\}_{N=1}^{\infty}$ bounded.

This proves part (1) of Comparison Test.

To see part (2), just notice that if $\{S_N\}_{N=1}^{\infty}$ is <u>not</u> bounded,

then $\{S_N'\}_{N=1}^{\infty}$ is not bounded either.

This proves part (2).

## Limit Comparison Test

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms.
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ is a finite number with $c > 0$,
then either both series converge or the both diverge.

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## Examples

1. $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ compare to $\sum \frac{1}{2^n}$, geom. series with $r=\frac{1}{2}$.
$\Rightarrow$ it converges.

Let $a_n = \frac{1}{2^n-1}$; $b_n = \frac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/2^n-1}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0$$

By limit comparison test, since $\sum \frac{1}{2^n}$ Converges, so does $\sum \frac{1}{2^n-1}$.

2. $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ compare to $\sum \frac{n^2}{n^{5/2}} = \sum \frac{1}{n^{1/2}}$ p-series with $p=\frac{1}{2}$
so it diverges.

$$\lim_{n\to\infty} \left( \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{1/n^{1/2}} \right) = \lim_{n\to\infty} \left( \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot n^{1/2} \right)$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2+3/n}{\sqrt{5/n^5+1}} = 2 > 0$$

Since $\sum \frac{1}{n^{1/2}}$ diverges, we conclude that $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ also diverges.