11.5 Alternating Series

An <u>alternating series</u> is a series whose terms alternate between positive and negative.

<u>Example</u>
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ ← <u>alternating harmonic series</u>
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ ← <u>alternating p-series w/ p=2</u>
- $1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\dots$ <u>NOT</u> alternating.

<u>General Form</u> : $\boxed{\pm\sum_{n=1}^{\infty}(-1)^{n-1}b_n}$ with $b_n>0$ for all n.

<u>Alternating Series Test</u> (aka Leibnitz criterion)

Given an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ with $b_n>0$, satisfying (1) $b_{n+1}\le b_n$ for all n

(2) $\lim_{n\to\infty} b_n = 0$

then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ <u>converges</u>.

<u>Example</u> The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges since it satisfies the conditions (1) and (2) in the alternating series test.

Explanation

Let $S_N = \sum_{n=1}^{N} (-1)^{n-1} b_n$ be the Nth partial sum

- If $N = 2k$ is even, then

$$S_{2k} = (b_1 - b_2) + (b_3 - b_4) + \dots + (b_{2k-1} - b_{2k})$$

So $\{S_{2k}\}_{k=1}^{\infty}$ is monotone increasing and bounded above by b_1.

- If $N = 2k+1$ is odd, then

$$S_{2k+1} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots - (b_{2k} - b_{2k+1})$$

So $\{S_{2k+1}\}_{k=1}^{\infty}$ is monotone decreasing and bounded below by $b_1 - b_2$.

Therefore both $\{S_{2k}\}$ and $\{S_{2k+1}\}$ converge.

Moreore $\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} (S_{2k} + b_{2k+1})$

$$= \lim_{k\to\infty} S_{2k} + \cancel{\lim_{k\to\infty} b_{2k+1}}^{\,0} \quad \text{by hypothesis}$$

$$= \lim_{k\to\infty} S_{2k}.$$

Thus $\{S_k\}_{k=1}^{\infty}$ converges

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

Example $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^3+1}$

Q: Can we apply the Alternating Series Test?

- series is alternating ✓
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{1}{n+1/n^2} = 0$ ✓

But... is it true that $b_{n+1} \le b_n$ for all n? Why?

Consider function $f(x) = \frac{x^2}{x^3+1}$ on interval $[1,\infty)$.

Then one can show (Quotient Rule!) that $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$

Notice that $f'(x) < 0$ for $x > \sqrt[3]{2}$.

Therefore $f(x)$ is <u>decreasing</u> on $[\sqrt[3]{2}, \infty)$.

Therefore $\underset{\substack{\parallel \\ b_{n+1}}}{f(n+1)} < \underset{\substack{\parallel \\ b_n}}{f(n)}$ for $n = 2, 3, 4, \ldots$

So alternating series test applies to show $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^3+1}$

Converges.

Error Estimate

For any convergent series $\sum_{n=1}^{\infty} a_n$, the partial sums $S_N = \sum_{n=1}^{N} a_n$ can be used to approximate the value of $\sum_{n=1}^{\infty} a_n$, but just how accurate is the estimate?

The error is measured by a remainder term R_n, defined by setting $R_N = |S - S_N|$, where $S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$

Remainder Estimate for alternating series

Given $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, with $b_n > 0$ and $\lim_{n\to\infty} b_n = 0$, and such that $b_{n+1} \le b_n$ for all n

then $R_n \le b_{n+1}$.

Example Find $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ accurate to three decimal places.

Remark: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so the sum above converges to $e^{-1} = \frac{1}{e}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 0! - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots$$

Since $\frac{1}{5040} = b_7 \le 0.0002$, S_6 is accurate to within 0.0002

NOTE: $S_6 = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$.

Example

How many terms are needed to estimate $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ to within 10^{-4}?

Want $|R_n| \le b_{n+1} = \frac{1}{(n+1)^2}$ to be less than 10^{-4}

i.e. $\frac{1}{(n+1)^2} \le 10^{-4}$

$(n+1)^2 \geq 10^4$

$\Rightarrow n+1 \geq \sqrt{10^4} = 10^2$

$\Rightarrow n \geq 100$