11.6 Absolute convergence and ratio and root tests

Defn A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 1. The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is not absolutely convergent.

2. The series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent since $|a_n| = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is a p-series with $p=2>1$.

Defn A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it converges but is not absolutely convergent.

For example, $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is conditionally convergent

Proposition If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Explanation: $0 \le a_n + |a_n| \le 2|a_n|$

Since $\sum |a_n|$ is convergent, limit laws imply $\sum 2|a_n|$ converges.

Therefore, by comparison test, we see that

$\sum (a_n + |a_n|)$ also converges.

Notice that $\sum (a_n + |a_n|) - \sum |a_n| = \sum a_n$.

But $\sum (a_n + |a_n|)$ and $\sum |a_n|$ both converge, so does $\sum a_n$. ▨

<u>Ratio Test</u> Suppose $\sum_{n=1}^{\infty} a_n$ is a series.

Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

If $L < 1$, then $\sum a_n$ is <u>absolutely convergent</u>.

If $L > 1$ or $L = \infty$, then $\sum a_n$ diverges.

If $L = 1$ or limit does not exist, the ratio test is <u>inconclusive</u>.

<u>Ex.</u> 1. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}\frac{(n+1)^2}{2^{n+1}}}{(-1)^n \frac{n^2}{2^n}} = -\frac{(n+1)^2}{n^2 \cdot 2}$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2 \cdot 2} = \frac{1}{2}$

By ratio test, since $L = \frac{1}{2} < 1$, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}$ is <u>absolutely convergent</u>.

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2. $\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}}{\frac{n^n}{2^n n!}} = \frac{(n+1)^{\cancel{n+1}\ n}}{\cancel{2^{n+1}}\,2\ \cancel{(n+1)!}\,(n+1)} \times \frac{\cancel{2^n}\ \cancel{n!}}{n^n}$$

$$= \frac{(n+1)^n}{2n^n} = \frac{1}{2}\left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{2}\left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^n$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= \frac{1}{2}\cdot e \approx \frac{1}{2}(2.7182\ldots) > 1$$

So by ratio test, this series diverges.

Root test

Consider the series $\sum_{n=1}^{\infty} a_n$.

Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (|a_n|)^{1/n} = L$.

- If $L<1$, then $\sum a_n$ is absolutely convergent
- If $L>1$ or $L=\infty$, then $\sum a_n$ diverges.
- If $L=1$ or limit does not exist, then the root test is inconclusive

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Example $\sum_{n=1}^{\infty} \left(\frac{3n+4}{5n+1}\right)^n$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{3n+4}{5n+1}\right)^n} = \frac{3n+4}{5n+1}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{3n+4}{5n+1} = \frac{3}{5} < 1$$

Therefore, by the root test, $\sum \left(\frac{3n+4}{5n+1}\right)^n$ is absolutely convergent

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Remarks 1. The ratio test is often useful for series involving factorials.

2. The root test is often useful for series $\sum a_n$ where $a_n = (b_n)^n$.