11.7 Strategy for testing series

Review of different methods for determining the convergence or divergence of a given series.

Ex 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$.

Use the divergence test, since $a_n = \frac{n-1}{2n+1}$ and
$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-1}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{1-1/n}{2+1/n}\right) = \frac{1}{2} \neq 0$,
the divergence test applies to show this series diverges.

Ex 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Focus on top order terms, which are $\frac{\sqrt{n^3}}{3n^3} = \frac{1}{3n^{3/2}}$
we compare to p-series with $p = 3/2$.
Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, since $p = \frac{3}{2} > 1$.

We use limit comparison test, noting that
if $a_n = \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \cdot n^{3/2}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n^3+1}}{3n^3+4n^2+2}\left(\frac{1/n^3}{1/n^3}\right)$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n^3}}{3+4/n+2/n^3} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$$

Therefore, by the limit comparison test, since $\sum \frac{1}{n^{3/2}}$ converges, so does $\displaystyle\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Ex 3 $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$

Notice that $a_n = n e^{-n^2}$, and if we set $f(x) = x e^{-x^2}$ we see $a_n = f(n)$ and $\displaystyle\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x e^{-x^2}\,dx$

can be used to determine convergence of this series.

$$\int_1^{\infty} x e^{-x^2}\,dx = \lim_{t\to\infty} \int_1^{t} x e^{-x^2}\,dx \qquad \left\{ \begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array} \right.$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-x^2} \right]_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2} e^{-t^2} + \frac{1}{2} e^{-1} \right)$$

$$= \frac{1}{2} \cdot \frac{1}{e} - \frac{1}{2} \left(\lim_{t\to\infty} e^{-t^2} \right)$$

$$= \frac{1}{2e} - \frac{1}{2}(0) = \frac{1}{2e}$$

Since $\displaystyle\int_1^{\infty} x e^{-x^2}\,dx$ converges as an improper int'l, then integral test tells us $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$ converges too.

Ex 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

This is an alternating series. $(-1)^n a_n$ where $a_n = \dfrac{n^3}{n^4+1}$

The terms a_n are monotone decreasing, which follows by showing $f(x) = \dfrac{x^3}{x^4+1}$ has $f'(x) < 0$ for $x \geq 1$.

Indeed, $f'(x) = \dfrac{-x^6 - 3x^2}{(x^4+1)^2}$ (by ~~Chain~~ Quotient Rule)

and it is easily seen that $f'(x) < 0$ for $x \geq 1$

Therefore, since $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{n^4+1} = 0$, this series converges by alternating series test.

Ex 5 $\quad \displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

We use the ratio test, so we compute

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{2^{k+1}/(k+1)!}{2^k/k!}$$

$$= \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k}$$

$$= \lim_{k\to\infty} \frac{2}{k+1} = 0 < 1$$

So this series converges by the ratio test.

Ex 6 $\quad \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2+3^n)} \qquad 3^n < 2+3^n \Rightarrow \frac{1}{3^n} \geq \frac{1}{2+3^n}$

We use comparison test with geom. series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

Notice that the geom. series converges since it is $\displaystyle\sum_{n=1}^{\infty} r^n$ with $r = \frac{1}{3} < 1$.

Therefore, the <u>smaller</u> series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ must also converge.