11.8 Power Series

Defn A power series is one of the form

$$(*) \quad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

where x is a real variable and c_n's are coefficients.

You should think of the expression $(*)$ as "like a polynomial" but we will see it is different in many ways.

Example 1. (Geometric Series)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

As this is a geometric series in x, it will converge if $|x| < 1$ and diverge if $|x| \geq 1$.

Thus, for some values of x, the series $(*)$ will converge whereas for other values of x, it will diverge.

2. If $c_n = 0$ for all $n > N$ for fixed integer N, then the power series $(*)$ is a polynomial

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{N} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N$$

A series centered at a is one of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Notice the value of this series at $x = a$ is just c_0, the constant term.

<u>Basic Fact</u> Given the power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are three distinct possibilities:
(1) The series converges only for $x=a$
(2) The series converges for all x
(3) There exists $R>0$ such that the series converges absolutely if $|x-a|<R$ and diverges if $|x-a|>R$.

The <u>radius of convergence</u> is the number R in (3). If (1) holds, we set $R=0$, and if (2) holds, we set $R=\infty$.

The <u>interval of convergence</u> consists of all x for which the series converges. It includes the open interval $(a-R, a+R)$, together with possibly one or both endpoints.
Note: one must check $x=a-R$, $x=a+R$ <u>by hand</u>.

One can use the <u>ratio</u> <u>test</u> to determine the radius of convergence.

If $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$ exists and is finite (or ∞), then the radius of convergence for the series is

$$R = \begin{cases} 1/L & \text{if } 0<L<\infty \\ 0 & \text{if } L=\infty \\ \infty & \text{if } L=0 \end{cases}$$

<u>Remark</u> If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ does not exist, we get no information about the radius of convergence from the ratio test.

In that case, one can try using the root test instead of the ratio test. If the limit $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$ exists and is finite (or ∞), then again the radius of convergence is given by
$$R = \begin{cases} 1/L & \text{if } 0 < L < \infty \\ 0 & \text{if } L = \infty \\ \infty & \text{if } L = 0 \end{cases}$$

Example

1. $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + \cdots$

Recall convention that $0! = 1$

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty \Rightarrow R = 0$$

This series diverges except for $x = 0$.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{1/(n+1)!}{1/n!} \right|$$
$$= \lim_{n\to\infty} \left| \frac{1}{(n+1)!} \cdot \frac{n!}{1} \right|$$
$$= \lim_{n\to\infty} \left(\frac{1}{n+1} \right) = 0 \Rightarrow R = \infty$$

This series converges for all x.

Remark: We will see soon that $\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}$.

3 $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot 2^n}$$

This series is centered at $a=1$ and has coefficients $C_n = \frac{1}{n \cdot 2^n}$.

$$\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)\cdot 2^{n+1}}{1/n\cdot 2^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{1}{(n+1)\,2^{n+1}} \cdot \frac{n \cdot 2^n}{1}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n}{2(n+1)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{1}{2+2/n}\right| = \frac{1}{2} \Rightarrow R = 2$$

This series converges if $|x-1| < 2$
and diverges if $|x-1| > 2$.

Q: What about when $|x-1| = 2$?
We must check each of these two endpoints "by hand."
We know the series converges for x in the open interval $(-1, 3) = (a-R, a+R)$ $\begin{pmatrix} a=1 \\ R=2 \end{pmatrix}$
but we check convergence at $x=-1$ and $x=3$ separately.

If $x=3$, then $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(3-1)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$

divergent (harmonic series)

If $x=-1$, then $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1-1)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

convergent (by alternating series test).

<u>Remark</u> $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots = \ln(1/2)$ We will see why soon!