11.9 Functions as Power Series

(★) $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + \dots + u^n + \dots$

In (★) we have a geometric series in u, and on left is a rational function and on right is a power series.

This illustrates how different power series are from polynomials, since $\frac{1}{1-u}$ is not a polynomial in any way, shape or form!

We can use (★) to represent other functions as power series.

Ex 1. $\frac{1}{1+x^2}$

To make this look like (★), set $u = -x^2$, i.e. write

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + (-x^2)^2 + (-x^2)^3 + \dots$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Since (★) converges for $|u| < 1$ and diverges for $|u| \geq 1$ we see that $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges for $|-x^2| < 1$

and diverges for $|-x^2| \geq 1$.

So this series has radius of convergence $R = 1$ and interval of convergence $(-1, 1)$.

Remark: $|-x^2| < 1 \iff |x| < 1$.

Ex. 2 Find the power series representation for $\frac{1}{x+5}$, and determine its radius of convergence and its interval of convergence.

To accomplish this, we rewrite $\frac{1}{x+5}$ to look like ($\star$)

$$\frac{1}{x+5} = \frac{1}{5+x} = \frac{1}{5(1+\frac{x}{5})} = \frac{1}{5(1-(-\frac{x}{5}))} = \frac{1}{5}\left(\frac{1}{1-(-\frac{x}{5})}\right) \quad [u = \tfrac{-x}{5}]$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{-x}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^n}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{5^{n+1}}$$

Radius of convergence $\quad \left|-\frac{x}{5}\right| < 1 \iff |x| < 5 \Rightarrow R = 5$

Interval of convergence $\quad (-5, 5)$

[Note: this series diverges for $x = \pm 5$ - check!]

Ex. 3 $\quad \frac{x^2}{1+x}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n \qquad [u = -x]$$

So we can get $\frac{x^2}{1+x}$ by multiplying the series above by x^2.

$$\frac{x^2}{1+x} = x^2\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) = \sum_{n=0}^{\infty}(-1)^n x^{n+2} \qquad \text{reindex } m = n+2$$

so $n = m-2$

$$= \sum_{m=2}^{\infty}(-1)^{m-2}x^m \qquad \text{but } (-1)^{m-2} = (-1)^m$$

$$= \sum_{m=2}^{\infty}(-1)^m x^m = \sum_{n=2}^{\infty}(-1)^n x^n$$

$$= x^2 - x^3 + x^4 - x^5 + \cdots$$

Radius of convergence $\quad |x| < 1$ or $R = 1$

Interval of convergence $\quad (-1, 1) \quad$ [Series diverges if $x = \pm 1$.]

One can differentiate polynomials on a term-by-term basis, i.e.

If $p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N$

then $p'(x) = 0 + c_1 + c_2 \cdot 2 \cdot x + \dots + c_N \cdot N\, x^{N-1}$

Also, one can compute the integral $\int p(x)\,dx$ similarly, and $\int p(x)\,dx = k + c_0 x + c_1 \cdot \frac{1}{2} x^2 + c_2 \cdot \frac{1}{3} x^3 + \dots + c_N \frac{1}{(N+1)} x^{N+1}$

Here k = constant of integration.

The same method applies to power series, allowing us to find their derivatives and integrals as power series.

<u>Theorem</u> Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ with radius of convergence $R > 0$ ($R = \infty$ is allowed here), then the function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is continuous and differentiable on the open interval $(a-R, a+R)$ (all of $\mathbb{R}$ if $R = \infty$) with

$$f'(x) = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$$

$$\int f(x)\,dx = k + \sum_{n=0}^{\infty} c_n \cdot \frac{1}{(n+1)} (x-a)^{n+1} \qquad (k = \text{Const. of int.})$$

Both power series have radius of convergence R the same as the original power series. However, the interval of convergence for $f'(x)$ and $\int f(x)\,dx$ may change because of endpoint behavior.

Example Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We have already seen this series has radius of convergence $R = \infty$, i.e., it converges for all x.

Therefore, $f'(x) = \sum_{n=\cancel{0}}^{\infty} n \cdot \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ reindex using $m = n-1$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x).$$

Thus, we see that $f(x) = y$ is a solution to the differential equation $\frac{dy}{dx} = y$.

Furthermore, $y(0) = f(0) = 1$, so $f(x)$ is a solution to the initial value problem $y' = y$, $y(0) = 1$.

However, we have determined previously that $y = e^x$ is the unique solution to this initial value problem, therefore we conclude that $f(x) = e^x$, i.e.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Ex Find a power series representation for $\frac{1}{(1-x)^2}$ by differentiating ($\star$) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. What is its radius of convergence?

Clearly, if $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

then $f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}$

Differentiating the series $\sum_{n=0}^{\infty} x^n$ term by term, we see that

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1} \qquad m=n-1 \Rightarrow n=m+1$$

$$= \sum_{m=0}^{\infty} (m+1)\, x^m$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots$$

This series has radius of convergence $R=1$.

<u>Example</u> Using $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, together with the fact that $\int \left(\frac{1}{1+x}\right) dx = \ln|1+x| + C$ we can obtain a power series for $\ln(1+x)$ by integrating the series $\frac{1}{1+x}$ term by term.

Assume $-1 < x < 1$ in the following.

$$\ln(1+x) = \int \left(\frac{1}{1+x}\right) dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n\right) dx$$

$$= k + \sum_{n=0}^{\infty} \int (-1)^n x^n\, dx$$

$$= k + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad m=n+1$$

$$= k + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}$$

Plugging in $x=0$, $\ln(1+0) = \ln(1) = 0 \Rightarrow k=0$

Therefore

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Radius of convergence $R=1$.

Exercise: Show this power series converges at $x=1$ but not at $x=-1$.

Using the same idea, we can get a power series representation for $\tan^{-1}(x)$, using a power series for $\frac{1}{1+x^2}$ and integrating term by term.

Notice that $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

Also, $\tan^{-1}(x) = \int \frac{1}{1+x^2}dx + C$

Therefore
$$\tan^{-1}(x) = k + \int \sum_{n=0}^{\infty}(-1)^n x^{2n}$$
$$= k + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

However $\tan^{-1}(0) = 0 \Rightarrow k = 0$

Thus
$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

with radius of convergence $R=1$.

Exercise: check that this series converges for $x = \pm 1$.

(Leibniz) $\Rightarrow$
$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots$$

Slow convergence, as can be seen by estimating remainder $|R_n| < \frac{1}{2n+1}$

For instance, taking $n=10$, $S_{10} = \sum_{n=0}^{10}(-1)^n \frac{1}{2n+1} = 0.80807895\ldots$

accurate to $|R_{10}| < \frac{1}{23} = 0.043478\ldots$

Actual sum S satisfies $S_{11} < S < S_{10}$