Math 4E03 **Assignment 3** **due April 12, 2021**

In this assignment $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ the ring of integers and $\mathbb{F}_p$ the field of integers modulo p, where p is a prime number.[1]

1. Let $\mathbb{F}$ be a field and let α be an element of some algebraic extension of $\mathbb{F}$. Assume that $[\mathbb{F}(\alpha) : \mathbb{F}]$ is odd. Prove that $\mathbb{F}(\alpha) = \mathbb{F}(\alpha^2)$. (Hint: note that $\mathbb{F}(\alpha)$ is an extension of $\mathbb{F}(\alpha^2)$ and that α is a root of $x^2 - \alpha^2 \in \mathbb{F}(\alpha^2)[x]$.)

2. Let $\mathbb{K}$ be a finite extension of the field $\mathbb{F}$. Let $p(x) \in \mathbb{F}[x]$ be irreducible over $\mathbb{F}$ and assume that the degree of $p(x)$ does not divide $[\mathbb{K} : \mathbb{F}]$. Prove that $p(x)$ has no roots in $\mathbb{K}$. Use this to show that $x^2 - 11$ has no roots in $\mathbb{Q}(\sqrt[3]{3})$.

3. Let K be a finite extension of the field $\mathbb{F}$. Let $D \subset K$ be a subring of K (the ring operations of D are inherited from $\mathbb{K}$) such that $\mathbb{F} \subset D$. Show that D is a field, that is, show that every non-zero element of D has multiplicative inverse in D.

4. Find the splitting field over $\mathbb{Q}$ of the polynomial $x^6 - 9$. What is its degree over $\mathbb{Q}$?

5. Let n be a positive integer and assume $n = p^k m$ where p is prime and p does not divide m. Show that there are exactly m distinct n-th roots of unity over any field $\mathbb{F}$ of characteristic p. (Note that these roots of unity lie in an algebraic closure of $\mathbb{F}$.)

6. Let $f(x) = (x^2 - 2)(x^2 - 3)(x^2 - 11)$. Find the Galois group of $f(x)$ over $\mathbb{Q}$ and determine all subfields of its splitting field.

7. Let $\zeta = e^{2\pi i/5}$ (a primitive 5-th root of unity). Describe the Galois group of the polynomial $x^5 - 3 \in \mathbb{Q}(\zeta)[x]$ over $\mathbb{Q}(\zeta)$.

8. Show that $\mathbb{Q}\sqrt{2 + \sqrt{2}})$ is a Galois extension of degree 4 with cyclic Galois group.

[1] The assignment is not due until after Test #2. However, problems 1–5 concern material that is covered on Test 2. The recommendation is that you work out those five problems before Test #2.