

Math 4E03

Review problems on group theory

1. Answer true or false:

- (a) The alternating group A_7 is simple.
- (b) Any finite group of order 165 has a normal subgroup of order 5.
- (c) If p is prime, then any finite group of order p^2 is abelian.
- (d) If p is prime, then any finite group of order p^2 is cyclic.
- (e) If G is finite and H is a subgroup with $[G : H] = 3$, then H is characteristic.
- (f) The alternating group A_5 is generated by 3-cycles.

2. Let $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ be the quaternion group, with multiplication law as explained in section 1.5 of Dummit-Foote (or see lecture notes from Feb 10 and discussion of the division ring given by the quaternions \mathbb{H}). Note that Q_8 is a nonabelian group of order 8.

- (a) Is Q_8 simple? Why or why not? Justify your answer.
- (b) Is Q_8 perfect? Why or why not? Justify your answer.

3. (a) Write the element $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 7 & 6 & 5 & 1 \end{pmatrix}$ in S_7 as a product of disjoint cycles.

What is its order?

- (b) What is the order of its centralizer subgroup $C_G(\sigma)$, where G is the group S_7 ?
- (c) Does σ lie in A_7 ? Why or why not?

- 4. (a) Show that a subgroup of a solvable group is solvable.
- (b) Show that a quotient group of a solvable group is solvable.

5. Show that if G is a finite group of order pq , where p, q are distinct primes, then G is cyclic.

(If this is too difficult, try to prove it for $p = 3$ and $q = 5$.)