Basic Questions

(a) When are two knots (or links) equivalent?

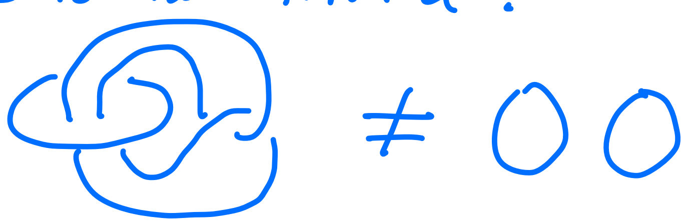
(b) When are two knots (or links) inequivalent?

Strategy on (a) is to search for a finite sequence of Reidemeister moves to relate the two knots.

On (b), the idea is to come up with invariants that show no such sequence of RMs exist. $\lambda(K_1) \neq \lambda(K_2)$

Simple question

1. How to show the trefoil 3_1 =  is not equivalent to the unknot?
2. How do we know that the Whitehead link is non-trivial?



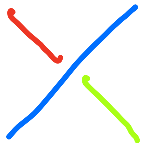
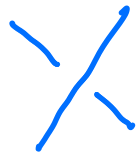
First knot invariant: tricolorability

We say a knot diagram is tricolorable if we can color the arcs using 3 colors $\{R, B, G\}$ such that

(i) At least 2 distinct colors are used

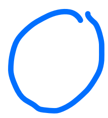
(ii) At any crossing, either all 3

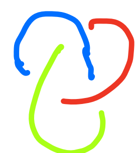
colors appear or all arcs have same color.

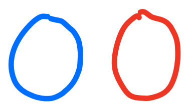
 or 

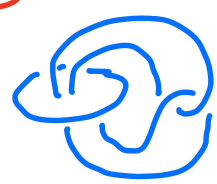
up to permutation of colors.

<u>Example</u>

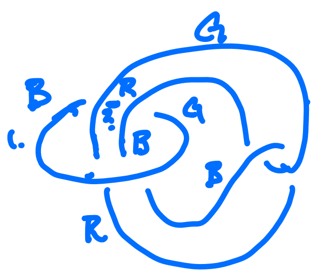
•  is not tricolorable. [(i) fails]

•  is tricolorable 

•  is tricolorable

but  is not.

To see why not, consider the two cases

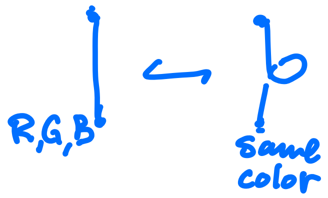


$G \neq B$ contradiction

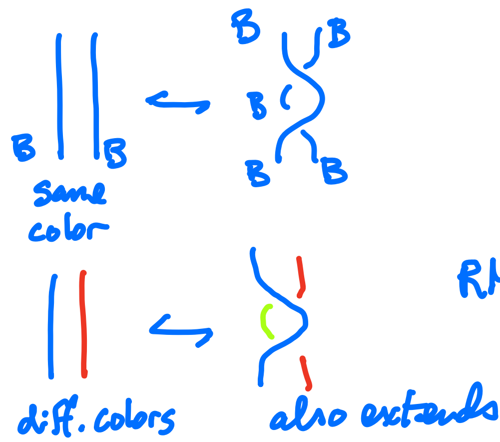
Also get contradiction if the first two arcs have same color

<u>Lemma</u> If two diagrams D_1 and D_2 are related by Reidemeister moves, then D_1 is tricolorable iff D_2 is.

Proof: (sketch)



RM I



RM II

RM III is more complicated
with more cases

$\Rightarrow$ KNOT INVARIANT

Applies to show $3_1 \neq 0$

It is a binary invariant and not so powerful, however, the same considerations show that the number of tricolorings is a knot invariant.

Mod p labels

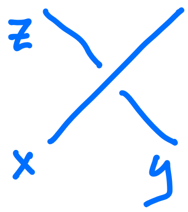
Replace colors R, B, G with numbers $\{0,1,2\}$ and extend this idea to sets $\{0,1,\dots,p-1\}$ where p is an odd prime.

We say a knot diagram has a mod p labeling if we can assign numbers from $\{0,1,\dots,p-1\}$ to each arc so that:

(i) At least two labels appear.

(ii) At each crossing, the labels

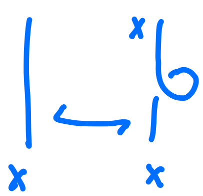
satisfy an equation:

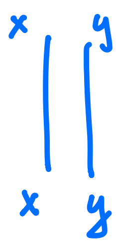


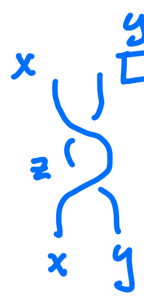
$\longleftrightarrow \quad 2x - y - z \equiv 0 \bmod p.$

<u>Theorem</u> If D_1 and D_2 are two knot diagrams and are related by RMs, then D_1 admits a mod p labeling iff D_2 does.

<u>Proof</u>



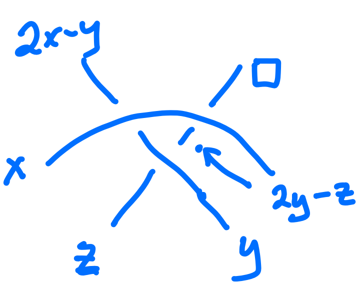


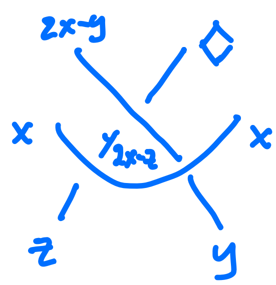


$\square = 2x - z \bmod p$

$= 2x-(2x-y) = y \bmod p$

$z \equiv 2x - y \bmod p$

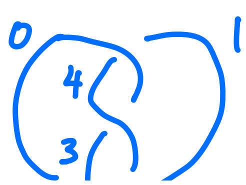




$$\square = 2x - (2y - z) = 2x - 2y + z$$

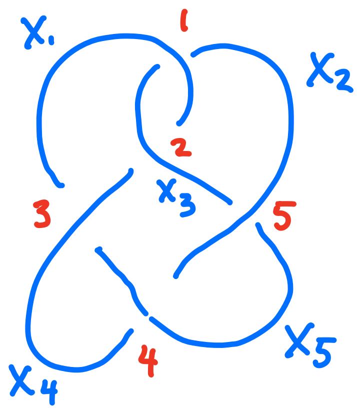
$$\lozenge = 2(2x-y) - (2x-z) = 4x - 2y - 2x + z = 2x - 2y + z$$ ∎

<u>Ex</u> 4_1 is mod 5 labelable.



One can use simple methods from linear algebra to address the question: Does a given knot admit a mod p labeling.

<u>Example</u> 5_2



<u>Step 1</u>: label all arcs with variables $x_1, \dots, x_5$

<u>Step 2</u>: Number the crossings $1, \dots, 5$

1: $2x_1 - x_2 - x_3$

2: $2x_3 - x_1 - x_4$

We can record the 5 equations using a matrix:

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 2 & -1 & 0 & -1 \end{bmatrix}$$

5×5 matrix

$i^{\underline{th}}$ row of A corresponds to $i^{\underline{th}}$ crossing

A mod p labeling is an element $(x_1, \dots, x_5)$ in $(\mathbb{Z}/p)^5$ in the kernel of A, and not all the x_i are equal.

- Observe that we could set $x_i = 1$ for $i = 1, \dots, 5$ and $(1, 1, \dots, 1)$ lies in the kernel of A.
- Also $\ker A$ is a subspace of $(\mathbb{F}_p)^5$ so if $(x_1, \dots, x_5)$ is a nonconstant solution, then so is $(x_1 - x_5, \dots, x_4 - x_5, 0)$

Conversely, if we can find a nontrivial solution $(x_1, \dots, x_{n-1}, 0)$ in $\ker A$, then the diagram is mod p labelable.

So we can remove the last column from A and consider the 5×4 matrix and its kernel. Further, the rows of this matrix are not lin. ind., in fact any row is a linear combination of the others.

So we can remove any row from A without affecting its kernel.

Let $\hat{A}$ be the 4×4 matrix obtained by removing the last row and column from A.

<u>Theorem</u> If K is a knot with n crossings, then this method produces an $n \times n$ matrix A defined over $\mathbb{Z}$ s.t. if $\hat{A}$ be the square $(n-1) \times (n-1)$ matrix obtained by removing a row and column from A, then K can be mod p labeled

$\iff$ the matrix $\hat{A}$ has a non-trivial solution mod p.

Remarks

1. There is a solution $\iff \det \hat{A} \equiv 0 \bmod p$

   i.e. if $p \mid \det \hat{A}$

2. The number of solutions is determined by the mod p nullity of $\hat{A}$.

$$\dim_{\mathbb{F}_p} \ker(\hat{A} : \mathbb{F}_p^{n-1} \to \mathbb{F}_p^{n-1})$$

$\mathbb{F}_p = \mathbb{Z}/p$ is the field of p elements

Defn
- $\det(K) = |\det(\hat{A})|$
- mod p rank $(K) =$ mod p nullity of $\hat{A}$

Theorem These two quantities are knot invariants, i.e. they do not depend on choices made (ordering of crossings) and are unchanged under RMs.