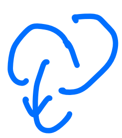
Lecture 6

Conway's Polynomial

L_+ $\quad$ L_- $\quad$ L_0

If L_+, L_-, L_0 are 3 links that are identical outside a small disk, where the appear as pictured above, then the Conway polynomial is the unique polynomial invariant of oriented links satisfying:

- $\nabla_O(z) = 1$ for $O =$ unknot
- $$\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\,\nabla_{L_0}(z)$$

Ex.  $\nabla_K(z) = z^2+1$ for trefoil.

FACT: If we substitute $z = t^{1/2} - t^{-1/2}$ into the Conway polynomial, we obtain the Alexander polynomial in "symmetrized" form.

$$\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$$

Consequently, if we normalize the Alexander polynomial so that (i) $\Delta_K(1) = 1$ and (ii) $\Delta_K(t^{-1}) = \Delta_K(t)$

(This is always possible to do.)

<u>Example</u> For the trefoil, $\Delta_K(t) = t^2 - t + 1$
Conway's normalization is $\Delta_K(t) = t - 1 + t^{-1}$

When the Alexander polynomial is normalized this way, it satisfies the skein relation

$$(*) \quad \Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t)$$

<u>Remark</u> Any polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ satisfying $f(1) = \pm 1$ and $f(t^{-1}) \doteq f(t)$ occurs as the Alexander polynomial of some knot. [Seifert, Levine]

The Alexander polynomial is remarkably effective at distinguishing low-crossing knots. For knots with 9 or fewer crossings there are only 5 duplicate pairs, i.e. $\Delta_{K_1}(t) = \Delta_{K_2}(t)$ for K_1, K_2 among the pairs:

$6_1, 9_{46}$
$7_4, 9_2$
$8_{14}, 9_8$
$8_{18}, 9_{24}$
$9_{28}, 9_{29}$

In the early 1980s, V. Jones introduced a new polynomial invariant of knots and links that completely revolutionized the field. The polynomial grew out of his work on von Neumann algebras. This discovery gave birth to a new field called <u>quantum topology</u>, and for it he was awarded a Fields medal in 1990.

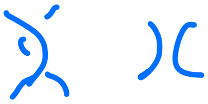
They are striking similarities between the Jones polynomial and the Alexander-Conway polynomial, but as invariants of knots they carry different information. Despite much research over the past 30-35 years, there are still many unanswered questions about the Jones polynomial.

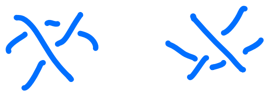
Jones' original definition was more complicated, using as it did representations of braid groups and quantum groups, but we will take a more down-to-earth approach and derive the Jones polynomial from the <u>Kauffman</u> <u>bracket</u>.

<u>FRAMED KNOTS</u>

<u>Definition</u> A framed knot (or link) is an equivalence class of oriented knot diagrams

modulo the "framed" Reidemeister moves:

(ii)  (same RM II)

(iii)  (same RM III)

(i')  (not the same as RM I !)

<u>Remark</u> Given a ^oriented^ knot diagram, define the local writhe of a crossing as


$w=+1$


$w=-1$

. The total writhe of a knot diagram is the sum of the local writhes over all crossings.

<u>Remark</u> Under a RM I, the total writhe changes by ± 1.

<u>Exercise</u> Show that the total writhe is an invariant of framed knots.

<u>Remark</u> The total writhe is <u>not</u> an invariant of (unframed) knots.

<u>Kauffman bracket</u>

Given a knot or link diagram K, we define a polynomial that is an invariant of framed knots.

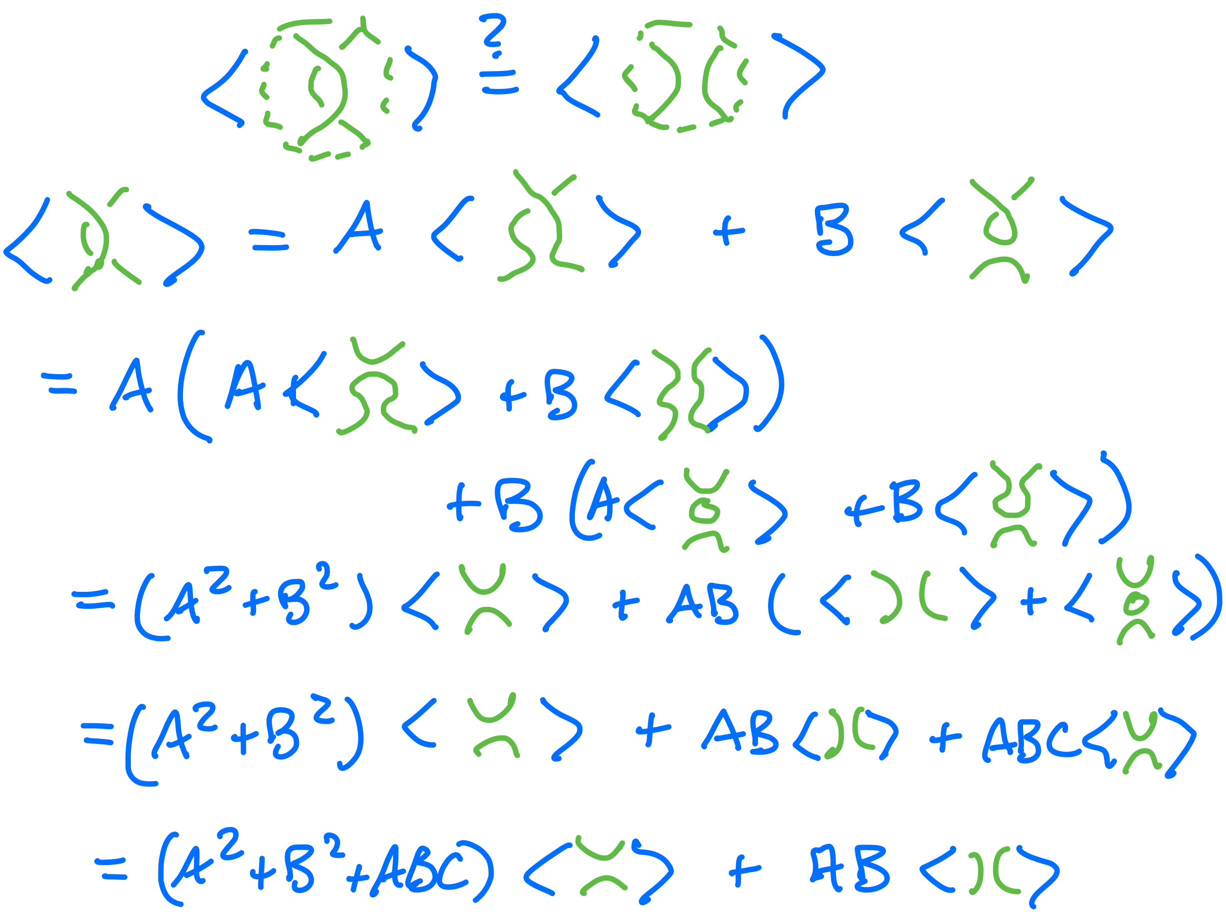
RULE 1 $\langle O \rangle = 1$ for unknot

RULE 2 

RULE 3 $\langle L \cup O \rangle = C \langle L \rangle$

We will determine values of A, B, C for which $\langle\ \rangle$ gives a well-defined invariant.

To start, consider the behavior of the bracket under RM II.

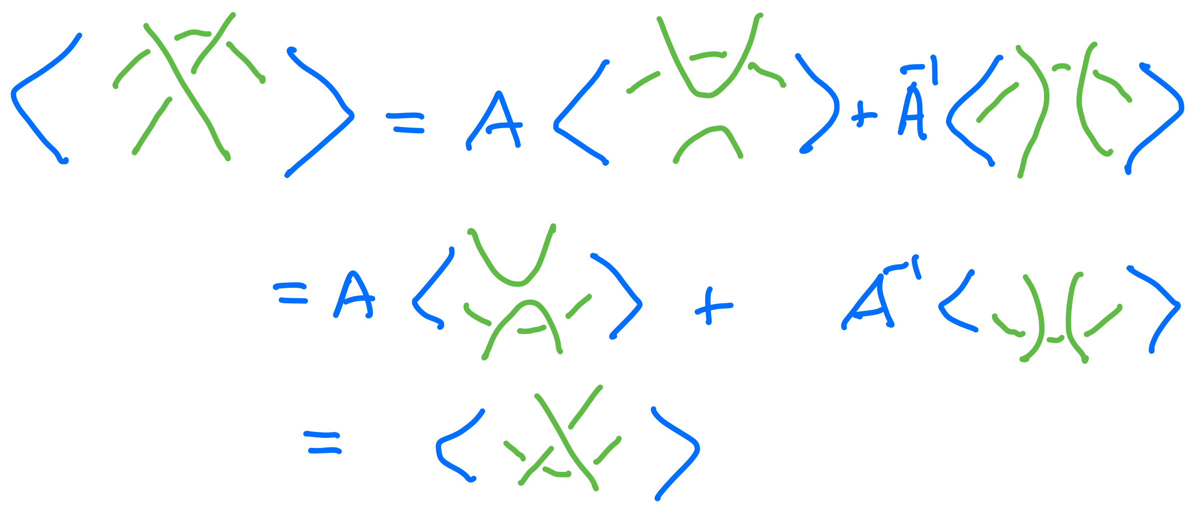


So we set $B = A^{-1}$

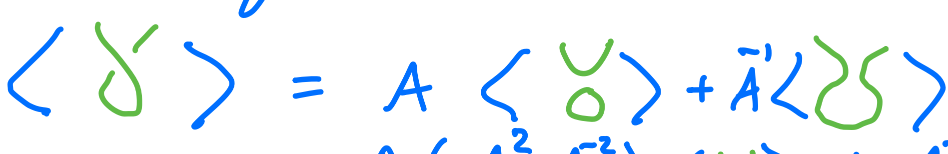
$A^2 + A^{-2} + C = 0 \Rightarrow C = -A^2 - A^{-2}$

With these values for B, C, the Kauffman bracket is invariant under RM II.

It turns out that with these values of B, C the Kauffman bracket is also invariant under RM III.



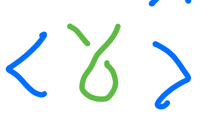
The Kauffman bracket is not invariant under RM I. In fact



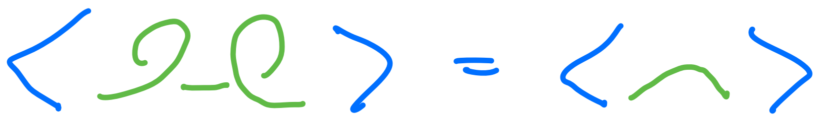
$$= A(-A^2 - A^{-2})\langle \cup \rangle + A^{-1}\langle \cup \rangle$$

$$= (-A^3 - A^{-1} + A^{-1})\langle \cup \rangle$$

$$= -A^3 \langle \cup \rangle$$

Similarly,  $= -A^{-3}\langle \cup \rangle$

<u>Exercise</u> Verify that the bracket is invariant under the modified RMI, i.e. that

 and

The Kauffman bracket is an invariant of framed unoriented links.

Although the bracket is not invariant under RMI, we can incorporate a correction term that will adjust for this.
The correction term is given by the total writhe of the diagram.

<u>Defn</u> The f-polynomial of an oriented link diagram is defined in terms of the Kauffman bracket by setting

$$f_K(A) = (-A^{-3})^{w(K)} \langle K \rangle$$

<u>Lemma</u> $f_K(A)$ is an invariant of oriented links.

<u>Proof</u>: Both the total writhe $w(K)$ and the Kauffman bracket $\langle\ \rangle$ are invariant

under RM II & III. So all we need to do is show invariance under RM I.

K

K'

$$f_K(A) = (-A^{-3})^{w(K)} \langle K \rangle$$
$$f_{K'}(A) = (-A^{-3})^{w(K')} \langle K' \rangle$$

But $\quad w(K) = w(K') - 1$

and $\quad \langle K \rangle = -A^{-3} \langle K' \rangle$

$$\begin{aligned} f_K(A) &= (-A^{-3})^{w(K)} \langle K \rangle \\ &= (-A^{-3})^{w(K')-1} \langle K \rangle \\ &= (-A^{-3})^{w(K')} (-A^{3}) \left[-A^{-3} \langle K' \rangle \right] \\ &= (-A^{-3})^{w(K')} \langle K' \rangle = f_{K'}(A). \ \blacksquare \end{aligned}$$