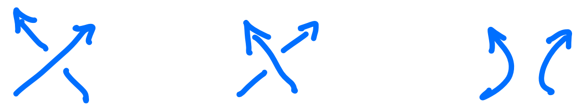
Lecture 7

Conway-Alexander polynomial is the unique polynomial of oriented links such that

- $\nabla_{U}(z) = 1$ for U = unknot
- $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\, \nabla_{L_0}(z)$

where



L_+ L_- L_0

Remark: • The invariant $\nabla_L(z)$ can be computed in terms of a skein resolution of L.

• The Conway potential $\nabla_L(z)$ is related to the Alexander polynomial by setting $z = t^{1/2} - t^{-1/2}$.

i.e. $\Delta_K(t) = \nabla_K(z = t^{1/2} - t^{-1/2})$

In that normalization, $\Delta_K(1) = 1$ and $\Delta_K(t^{-1}) = \Delta_K(t)$.

• Recall that $\Delta_K(t)$ does not distinguish a knot K from its mirror image. It also does not distinguish 6_1 from 9_{46}.

Both have $\Delta_K(t) = 2 - 5t + 2t^2$

$\doteq 2t^{-1} - 5 + 2t$

The Jones polynomial represents a vast improvement over the Alexander-Conway poly.

The Jones polynomial is defined in terms of the Kauffman bracket.

The bracket $\langle K \rangle$ is defined for knot and link diagrams and satisfies

- $\langle O \rangle = 1$
- $\langle L \cup O \rangle = (-A^2 - A^{-2}) \langle L \rangle$
- $\langle$  $\rangle = A \langle)(\rangle + A^{-1} \langle$  $\rangle$

$\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ and it is invariant under RM II & III moves, but not RM I.

Lemma $\langle$  $\rangle = -A^{3} \langle$  $\rangle$ and $\langle$  $\rangle = -A^{-3} \langle$  $\rangle$

Corollary $\langle$  $\rangle = -A^3 \langle \cup \rangle = -A^3$

$\langle$  $\rangle = -A^{-3} \langle \cup \rangle = -A^{-3}$

Defn The f-polynomial of an oriented link is the quantity

$$f_K(A) = (-A^{-3})^{w(K)} \langle K \rangle$$

where $w(K)$ = total writhe of K.
It is invariant under all 3 Reidemeister moves.

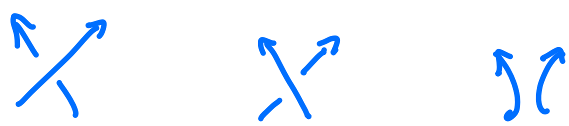
Defn The Jones polynomial is obtained by setting $t^{-1/2} = A^2$ in the f-polynomial.

i.e. $V_K(t) = f_K(t^{-1/4} = A)$.

Theorem (Jones) There exists a unique polynomial invariant of oriented knots and links satisfying ① $V_{\mathcal{U}}(t) = 1$ for $\mathcal{U}$ = unknot

② $t^{-1} V_{L_+}(t) - t\, V_{L_-}(t) = (t^{1/2} - t^{-1/2})\, V_{L_0}(t)$

for



L_+ $\quad L_-$ $\quad L_0$

$V_L(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$

Proof (2) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

$\langle \times \rangle = A^{-1} \langle)(\rangle + A \langle \asymp \rangle$

$A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle)(\rangle$

$\Rightarrow A \langle L_+ \rangle - A^{-1} \langle L_- \rangle = (A^2 - A^{-2}) \langle L_0 \rangle$

For oriented links, notice that

$w(L_+) = w(L_0) + 1$

$w(L_-) = w(L_0) - 1$

Therefore, $f_{L_+}(A) = (-A^{-3})^{\omega(L_+)}\langle L_+\rangle =$

$$=(-A^{-3})^{\omega(L_0)}\cdot(-A^{-3})\langle L_+\rangle$$

$$f_{L_-}(A) = (-A^{-3})^{\omega(L_-)}\langle L_-\rangle$$

$$= (-A^{-3})^{\omega(L_0)}(-A^{3})\langle L_-\rangle$$

$$A^4 f_{L_+}(A) - A^{-4} f_{L_-}(A)$$

$$= (-A^{-3})^{\omega(L_0)}\left[-A\langle L_+\rangle + A^{-1}\langle L_-\rangle\right]$$

$$= (-A^{-3})^{\omega(L_0)}\left[(-A^{2}+A^{-2})\langle L_0\rangle\right]$$

$$\Rightarrow \quad t^{-1} V_{L_+}(t) - t\, V_{L_-}(t) = (-t^{-1/2}+t^{1/2})\, V_{L_0}(t) \qquad \blacksquare$$

<u>Examples</u> 1. $\langle \bigcirc\ \bigcirc \rangle = (-A^{2}-A^{-2})\langle \bigcirc \rangle$

$$= -A^{2}-A^{-2}$$

$$V_{\bigcirc\bigcirc}(t) = -t^{-1/2} - t^{1/2}$$

2. Hopf links

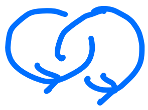
$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= A(-A^{3}) + A^{-1}(-A^{-3})$ by Cor.

$= -A^{4} - A^{-4}$

Now consider the two Hopf links

 H_+ positive Hopf link $\omega(H_+) = 2$

 H_- negative Hopf link $w(H_-) = -2$

$$V_{H_+}(t) = \left[(-A^{-3})^2 \langle H \rangle\right]_{t^{-1/4}=A}$$

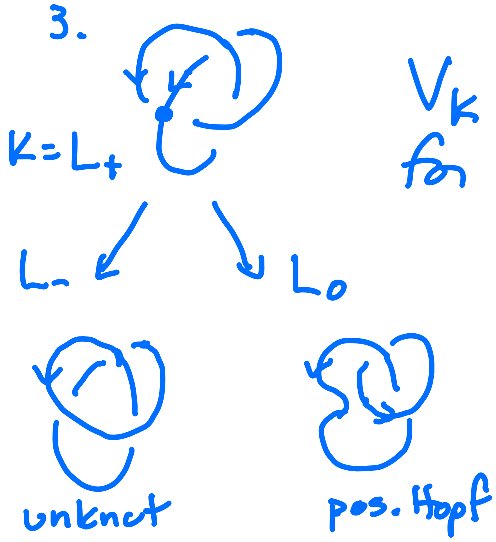
$$= A^{-6}(-A^{4} - A^{-4})\Big|_{t^{-1/4}=A}$$

$$= -A^{-2} - A^{-10}\Big|_{t^{-1/4}=A} = -t^{1/2} - t^{5/2}$$

Likewise $V_{H_-}(t) = -t^{-1/2} - t^{-5/2}$

(Notice that H_- is the mirror image of H_+, and under taking mirror images, the Jones polynomials are related by replacing t by t^{-1}.)

3.



$V_K(t) = -t^4 + t^3 + t$ for K = RH trefoil.

$$t^{-1} V_{L_+} - t V_{L_-} = (t^{1/2} - t^{-1/2}) V_{L_0}$$

$$t^{-1} V_K = t \cdot 1 + (t^{1/2} - t^{-1/2})(-t^{1/2} - t^{5/2})$$

$$= t + (-t - t^3 + 1 + t^2)$$

$$= -t^3 + 1 + t^2$$

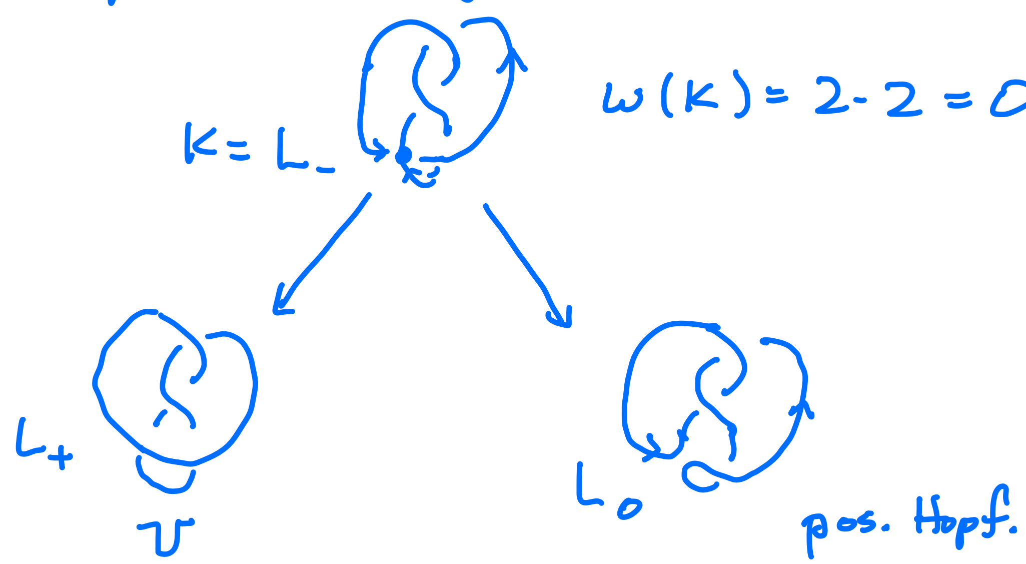
$$V_K(t) = -t^4 + t^3 + t$$

Notice that if K^* is the left hand trefoil, the mirror image of the right hand trefoil, then $V_{K^*}(t) = -t^{-4} + t^{-3} + t^{-1}$

$$V_K(t) \neq V_{K^*}(t)$$

$\Rightarrow$ The RH trefoil is not equivalent to the LH trefoil.

<u>Example</u> The figure 8 knot 4_1



$w(K) = 2 - 2 = 0$

$$t V_{L_-} = t^{-1} V_{L_+} - (t^{1/2} - t^{-1/2}) V_{L_0}$$

$$t V_K = t^{-1} \cdot 1 - (t^{1/2} - t^{-1/2})(-t^{5/2} - t^{1/2})$$

$$= t^{-1} - (-t^3 - t + t^2 + 1)$$

$$= t^{-1} + t^3 + t - t^2 - 1$$

$$tV_K = t^3 - t^2 + t - 1 + t^{-1}$$

$$V_K(t) = t^2 - t + 1 - t^{-1} + t^{-2}$$

Notice that $V_K(t^{-1}) = V_K(t)$.
This reflects the fact that the figure 8 knot is equal to its mirror image.
i.e. $K^* = K$ for $K = 4_1$

<u>Defn</u> A knot K is said to be <u>amphicheiral</u> if it is equivalent to its mirror image.

<u>Exercise</u> Show that the figure 8 knot is amphicheiral.

<u>Theorem</u> If K is amphicheiral then
$$V_K(t) = V_K(t^{-1})$$

The converse is <u>not</u> generally true. For example, 8_{17} is not amphicheiral even though its Jones polynomial is symmetric.

Remark 1. $V_{K^*}(t) = V_K(t^{-1})$

2. $V_U(t) = 1$ for U = unknot

OPEN PROBLEM

Does the Jones polynomial detect the unknot?

This has been verified for diagrams up to 25 crossings.

3. Determine necessary and sufficient conditions for a polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ to occur as the Jones polynomial for some knot $K \subseteq S^3$.

4. **Property** $V(K_1 \# K_2) = V(K_1) V(K_2)$