Lecture 8

We introduced the Kauffman bracket $\langle D \rangle$ and Jones polynomial $V_K(t)$.

The Kauffman bracket is an invariant of unoriented framed diagrams and $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$

The Jones polynomial is an invariant of oriented knots and links and $V_K \in \mathbb{Z}[t, t^{-1}]$

The two are related by

$$V_K(t) = \left[(-A)^{-3w(D)} \langle D \rangle \right]_{A = t^{-1/4}}$$

where D is a diagram for K.

The Jones polynomial satisfies

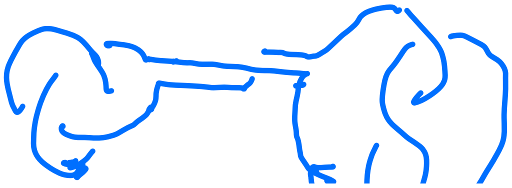
(i) $V_U(t) = 1$ for U = unknot

(ii) $V_{K \# J}(t) = V_K(t)\, V_J(t)$ for connected sums

(iii) $V_{K^*}(t) = V_K(t^{-1})$ where K^* is the mirror image of K.

Remarks.

- The connected sum is a well-defined operation on oriented knots.



K

J

- The mirror image of a knot is the one obtained by changing all the crossings.

We saw how to compute $\langle D \rangle$ and $V_K(t)$ using the skein relation, and while this method works in general it is sometimes cumbersome to implement.

There is an alternative formulation for computing $\langle D \rangle$ which shows it to be a well-defined function of unoriented framed knots (and links).

- $\langle \bigcirc \rangle = 1$
- $\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$
- $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

Lemma $\langle \overbrace{\bigcirc \sqcup \cdots \sqcup \bigcirc}^{n \text{ components}} \rangle = (-1)^{n-1}(A^2 + A^{-2})^{n-1}$

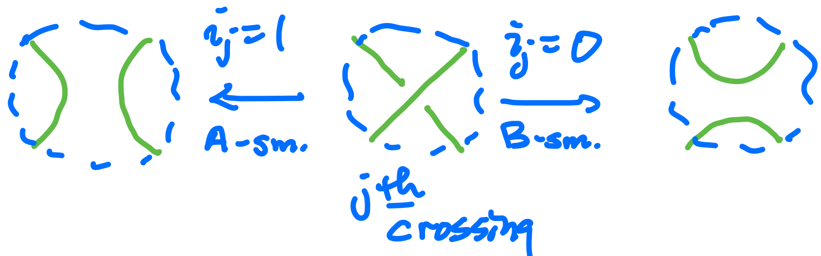
Proof Easy induction using 2nd property.

We can apply this Lemma to any link

diagram $\hat{D}$ without crossings to compute $\langle \hat{D} \rangle$ by simply counting the number of components of $\hat{D}$.

By applying the Kauffman relation to each crossing in a given diagram D, we can replace $\langle D \rangle$ by a linear combination with coefficients in $\mathbb{Z}[A, A^{-1}]$ with diagrams $\langle \hat{D}_I \rangle$, where each $\hat{D}_I$ is a diagram without crossings.

Here $I = (i_1, \dots, i_n)$ is a multi-index with $i_j = 0$ or 1.



If the diagram D has n crossings, then there are a total of 2^n smoothings.

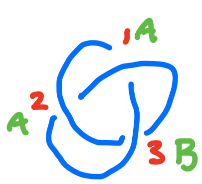
A <u>state</u> S is the result of performing smoothings at each crossing of D.

Let $a(S) = \#$ A smoothings in S

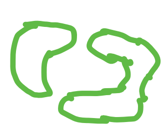
$b(S) = \#$ B smoothings in S

$|S| =$ number of components of S.

<u>Example</u>



AAB smoothing (110)



$a(S) = 2$, $b(S) = 1$

$|S| = 2$

$$\langle D \rangle = \sum_{S} A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1}$$

This is a sum over all 2^n smoothings of D.

<u>Applications of Jones polynomial</u>

The first applications of $V_K(t)$ to knot theory were toward detecting amphicheirality of knots and determining the crossing number of a knot.

For a knot K, the crossing number is the minimum, over all diagrams D for K, of the number of crossings of D.

<u>OPEN PROBLEM</u> Is the crossing number additive under connected sum?

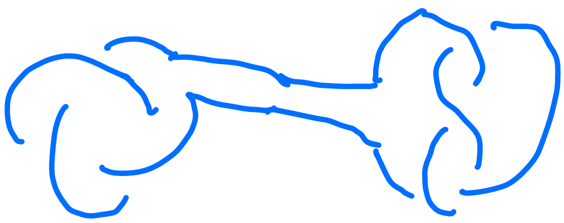
i.e. is $c(K \# J) = c(K) + c(J)$?

<u>Remark</u>:
- Let $c(K)$ = crossing number of a knot K. (notation)
- We know by construction that
$$c(K \# J) \leq c(K) + c(J).$$
Indeed, if D, E are diagrams for K, J respectively, then $D \# E$ is a diagram for $K \# J$.

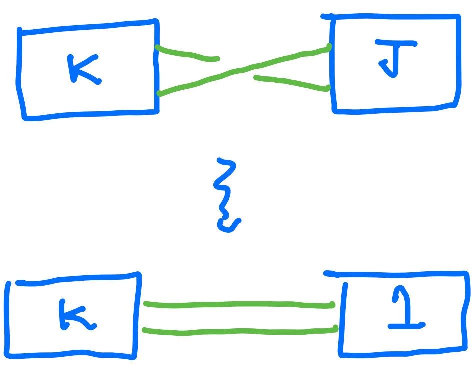
Defn A knot diagram is *alternating* if the crossings alternate between over and under crossings as one traverses the knot.

A knot is alternating if it can be represented by an alternating diagram.

Remark Not all knots are alternating. The first non-alternating knot in the tables is 8_{19}.



Notice if D, E are two alternating diagrams, then we can always construct the connected sum $D \# E$ to be an alternating diagram



The crossing on the left is a *nugatory* crossing. It is removable, and this decreases the crossing number.

Defn A knot or link diagram is said to be *reduced* if it does not contain a nugatory crossing.

Theorem 1 [Kauffman - Murasugi - Thistlethwaite] If the knot K has a reduced alternating diagram D with n crossings, then K has crossing number n.

In other words, any reduced alternating diagram has minimal crossing number.

Corollary If K and J are alternating knots, then $c(K \# J) = c(K) + c(J)$.

Proof of Corollary Let D, E be reduced alternating diagrams for K, J resp. Then Thm 1 implies $c(K) = c(D)$ and $c(J) = c(E)$. Further, observe that $D \# E$ is also reduced and alternating, or at least we can construct it to be, and then again by Thm 1 we have

$$c(K \# J) = c(D \# E) = c(D) + c(E) = c(K) + c(J)$$

It remains to prove Theorem 1. To do that we will use the Jones polynomial to obtain a bound on the crossing number.

<u>Defn</u> If $f \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial then set
$$\text{span}(f) = \text{maxdeg}_t(f) - \text{mindeg}_t(f).$$

<u>Example</u> $f = t + t^3 - t^4 \Rightarrow \text{span}(f) = 4 - 1 = 3$
$g = 1/t^2 - 1/t + 1 - t + t^2 \Rightarrow \text{span}(g) = 2 - (-2) = 4$

Note: f is the Jones polynomial of 3_1 and g is the Jones poly for 4_1

<u>Theorem 2</u> 1. For any knot diagram D with n crossings, $\text{span}(V_D) \leq n$.
2. If D is a reduced alternating diagram then $\text{span}(V_D) = n$.

<u>Remark</u> Note that Thm 1 is a consequence of Thm 2. To see this, we can argue by contradiction as follows.

Suppose K admits a reduced alternating diagram D, and suppose that D is <u>not</u> a minimal crossing diagram.

So K also admits a diagram D' with $c(D') < c(D)$.

Let $n = c(D)$

Then $\operatorname{span}(V_D) = n$

$\operatorname{span}(V_{D'}) \leq c(D') < n.$

But $V_K = V_D = V_{D'}$ is an invariant of K,

so $\operatorname{span} V_{D'} = \operatorname{span} V_D$ $\times$