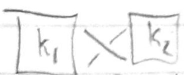
Lecture 10

KMT Theorem

Theorem 1: [KMT]

If K is a knot and is represented by a reduced alternating diagram D with n crossings, then K has crossing number $c(K)=n$.

Reduced = No nugatory crossings



Nugatory Crossing

Corollary:

If K and J are alternating knots then $c(K\#J)=c(K)+c(J)$

Hard Open Problem

Prove (or disprove) additivity of the crossing number.

Theorem 2:

1) If D is a knot diagram with n crossings then $\operatorname{span}(V_D)\leq n$
2) If D is a reduced alternating diagram with n crossings then $\operatorname{span}(V_D)=n$

Remark

Thm 2 $\Rightarrow$ Thm 1

To see this suppose K is a knot with a reduced alternating diagram D, then $\operatorname{span}(V_D)=n=c(D)$.

Suppose $c(K)<n$, then we have a diagram D' for K with $c(D')<n$. By 1. $\operatorname{span}(V_{D'})\leq c(D')<n$, But $V_{D'}=V_D=V_K$ (it is an invariant) $\divideontimes$

It remains to prove thm 2.

Feb 8

Proof of KMT

Proof of Theorem 2:

Using the Kauffman bracket

Recall $V_D(t) = \left(-A^{-3w(D)} \langle D \rangle\right)\Big|_{A = t^{-1/4}}$

Notice that $\text{span}_A(\langle D \rangle) = 4\,\text{span}_t(V_D)$

↳ with respect to the variable A

Thm 2 is equivalent to

1) $\text{span}(\langle D \rangle) \leq 4n$
2) $\text{span}(\langle D \rangle) = 4n$ if D reduced alternating

To prove this we will use the state-sum model

$$\langle D \rangle = \sum_{S} A^{a(S) - b(S)} (-A^2 - A^{-2})^{|S|-1}$$

↳ # of components of S

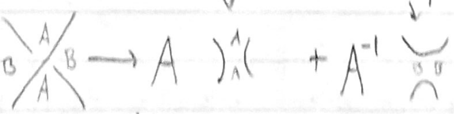
↳ 2^n different ways to smooth

# of A smoothings

# of B smoothings

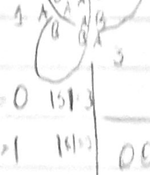
A splitting

B splitting



We apply this smoothing at each crossing.

e.g.



1 for A, 0 for B

S_A =

Splitting	Diagram			
111		$a(s)=3$	$b(s)=0$	$\lvert s\rvert=3$
110		$a(s)=2$	$b(s)=1$	$\lvert s\rvert=$[illegible]
101		$a(s)=2$	$b(s)=1$	$\lvert s\rvert=2$
011		$a(s)=2$	$b(s)=1$	$\lvert s\rvert=2$
100		$a(s)=1$	$b(s)=2$	$\lvert s\rvert=1$
010		$a(s)=1$	$b(s)=2$	$\lvert s\rvert=1$
001		$a(s)=1$	$b(s)=2$	$\lvert s\rvert=1$

000 ⓞ $a(s)=0$ $b(s)=3$ $|s|=2$

↳ S_B

Let S_A be the state with all A smoothings

Let S_B " " B smoothings

Proof of KMT & Dual State Lemma

Proof cont:

$a(S_A) = n \quad b(S_A) = 0$

$a(S_B) = 0 \quad b(S_B) = n$

One would expect S_A to give the term in $\langle D \rangle$ of maximal A-degree and S_B the term of minimal A-degree

Maximal Degree

$$M(S_A) = a(S_A) + 2|S_A| - 2$$
$$= n + 2|S_A| - 2$$

Minimal Degree

$$m(S_B) = -b(S_B) - 2|S_B| + 2$$
$$= -n - 2|S_B| + 2$$

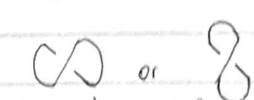
$$\text{span}(\langle D \rangle) \leq M(S_A) - m(S_B)$$
$$= n + 2|S_A| - 2 + n + 2|S_B| - 2$$
$$= 2n + 2|S_A| + 2|S_B| - 4$$

$\text{Span}(\langle D \rangle) \leq 2n + 2(n+2) - 4 = 4n$ (By dual state Lemma)

Lemma: Dual State Lemma

If D is a connected link diagram with n crossings then $|S_A| + |S_B| \leq n+2$

Proof by induction

Consider the case $n=1$  two possible connections 

in either case $|S_A| = 2$ and $|S_B| = 1$ or $|S_A| = 1$ and $|S_B| = 2$, so $|S_A| + |S_B| = 3 \leq 3 = n+2$

Assume its proved for all connected link diagrams D' with $n-1$ crossings.

Let D be a link diagram with n crossings. Choose a crossing and smooth it, for one of the smoothings the resulting diagram is connected. Let D' be the resulting diagram we can assume WLOG that D' is obtained by doing the A-smoothing. D' is connected with at most $n-1$ crossings so by induction (b/c with loss) we have

$$|S_A(D')| + |S_B(D')| \leq (n-1) + 2$$

Since $D \xrightarrow{A} D'$ $\underbrace{|S_A(D)| = |S_A(D')|}_{\text{All A smoothings}}$, $\underbrace{|S_B(D)|}_{\text{All B}} = \underbrace{|S_B(D')| + 1}_{\text{1 less B smoothing}}$

$$\Rightarrow |S_A(D)| + |S_B(D)| \leq n+2$$

Proof of KMT

Proof cont.

claim $\text{span}(\langle D \rangle) = 4n$ if D is red. alt.

General Fact:

Any knot or link diagram in S^2 admits a checker board coloring, ie colour the regions of $S^2 \setminus D$ with two colors so no two adjacent regions have the same color.

Notice that an A splitting joins the unshaded regions and B splittings join the shaded regions

Another Fact:

If D is a diagram with n crossings, this is a graph with n vertices and $2n$ edges and $(B+W)$ faces, $2 = \chi(S^2) = n - 2n + B + W \Rightarrow B + W = 2 + n$.

# of Black Regions

# of white

And when D is a reduced alternating diagram $|S_B| = B$, $|S_A| = W \Rightarrow \text{span}(\langle D \rangle) = 4n$

Consequences of the KMT

Theorem

Any two reduced alternating diagrams of the same knot have the same number of crossings

Recall,

A knot K is said to be amphicheiral if $K = K^*$ mirror image.

eg $3_1 \neq 3_1^*$ so 3_1 is not amphicheiral

$4_1 = 4_1^*$ so 4_1 is amphicheiral

Theorem

If K is a red. alt. knot with n crossings, with n odd, then K is not amphicheiral

Reason:

$\text{span}(V_K) = n$ is odd and $V_{K^*}(t) = V_K(t^{-1})$, the max deg term and min degree term have opposite parity. So the parity of $V_{K^*}(t)$ switches the parity of the max & min degree terms.