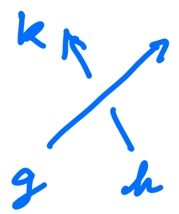
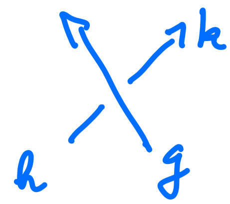
March 5, 2022

One can get useful invariants using knot labelings with the labels taken as elements in a group G.

1) <u>Consistency</u>



$k = g^{-1} h g$



$k = g h g^{-1}$

Here $g, h, k \in G$ are the labels on the arcs

2) <u>Generation</u>

The labels on the arcs that appear generate the whole group G.

<u>Theorem</u> If a knot diagram for K can be labeled using elements of a group G, then any equivalent knot diagram can also be labeled using elements of G.

<u>Conjugacy and labelings</u>

Recall elements $g, h \in G$ are <u>conjugate</u> if there is some element $x \in G$ with $g = xhx^{-1}$.

Ex $(12)(345)$ and $(35)(124)$ are conjugate elements in S_5.

Fact In the symmetry group S_n, two elements are conjugate if and only if they have the same cycle structure.

Theorem 2 If a knot diagram of an oriented knot can be labeled with elements from a conjugacy class of G, then any equivalent diagram can also be labeled with elements from that conjugacy class.

This allows for a more fine-tuned invariant of knots, one for each group G and choice of conjugacy class in G.

For a given group, it's natural to ask how many different invariants of knots you get. This is determined by how many different conjugacy classes you have in G.

Q: For the symmetric group S_n, how many conjugacy classes are there?

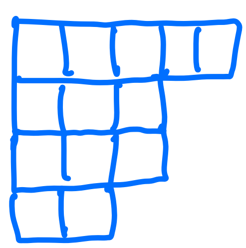
A: The number of partitions $p(n)$.

We can picture the conjugacy classes using Young tableaux.

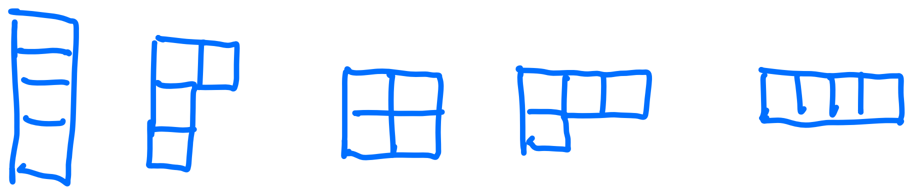
If we have a cycle $(i_1,\dots,i_{k_1})(i_{k_1+1},\dots i_{k_1+k_2})$
$\cdots(i_{k_1+\dots+k_{r-1}+1}\dots i_n)$
where $k_1+k_2+\dots+k_r=n$
then the partition is $k_1,\dots,k_r$.
We can reorder the cycle so that
$k_1 \geq k_2 \geq k_3 \geq \dots \geq k_r > 0$

The associated tableau has k_1 blocks in the first row, k_2 blocks in the 2^{nd}, and so on.

Eg. A cycle of type $(5,3,3,2)$ has tableau



<u>Exercise</u>: We saw that S_3 has 3 conj. classes, below are the tableaux for S_4. which has 5 conj. classes.



Q: How many conjugacy classes does S_5 and S_6 have?

A: 7 for S_5

10 for S_6

In general, S_n will have $p(n)$ conj. classes.

We say an element $(\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$ is a partition of n of length k if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

$$p(n) = \# \left\{ (\lambda_1, \dots, \lambda_k) \mid \lambda \text{ is a partition of } n \right\}$$

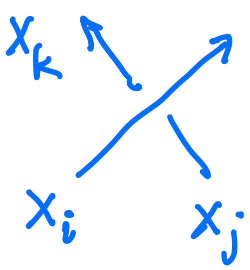
$$F(x) = \prod_{d=1}^{\infty} \frac{1}{(1-x^d)} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^d+x^{2d}+\cdots)\cdots$$

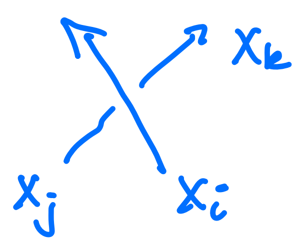
$$= 1 + \sum_{n=1}^{\infty} p(n)\, x^n$$

One can modify/refine these invariants and they turn out to be remarkably effective at distinguishing knots.

For example, Thistlethwaite used these invariants to compile a table of knots up to 13 crossings (12,965 prime knots). Note the Alex. poly. is not effective for this large class of knots, in fact 14 of them have Alex. poly. equal to 1.
[Eisermann, 2007, Pac. J. Math "Knot coloring polynomials"]

The knot labels can be replaced by group-valued variables, leading to an abstract finitely generated group associated to the knot K.





$x_k = x_i^{-1} x_j x_i$ $\qquad\qquad$ $x_k = x_i x_j x_i^{-1}$

Taking the labels $x_1, \ldots, x_n$ as the generators and using relations, one for each crossing as above, we get a <u>finite presentation</u> for a group G_K, called the <u>knot group</u>.

This is the Wirtinger presentation for G_K, and the isomorphism type of G_K is unchanged under Reidemeister moves.

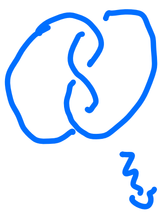
Exercise check this statement for RM1 + 2.

Theorem 3 The isomorphism type of the knot group G_K is invariant under Reidemeister moves and thus determines a group valued invariant of knots.

Remark: ① As an invariant of knots, the knot group is extremely powerful.
② It is generally difficult (or even impossible) to decide if two group presentations represent isomorphic groups.
③ One can even define a perfect invariant of knots using G_K provided one keeps track of the knot's peripheral structure. [T.B.Defined ... meridian and longitude]
④ There are examples of inequivalent knots K, K' with isomorphic groups, i.e. $G_{K_1} \cong G_{K_2}$. Ex: Square knot and granny

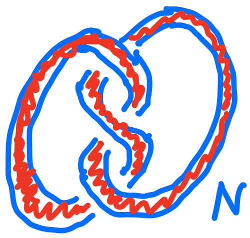
i.e. $3_1 \# 3_1$ and $3_1 \# 3_1^*$

(5) The knot group is the fundamental group of the complement of K in S^3.



K

If $K \subseteq S^3$ is a knot and $N(K)$ is a tubular neighborhood of K.



N(K)

remove red portion from S^3

Set $X_K = S^3 \setminus \mathring{N}(K)$

$G_K = \pi_1(X_K, *)$

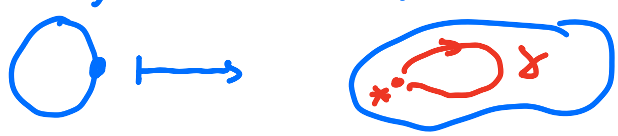
fund. group

consists of homotopy classes of loops in X_K based at $*$.

X - top. space

$*$ - basepoint in X

A loop $\gamma : (S^1, 1) \longrightarrow (X, *)$



If γ, γ' are two loops in X based at $*$, a homotopy between them is a cont. map $H : S^1 \times I \longrightarrow X$

s.t. $\left.\begin{array}{l} H(x,0) = \gamma(x) \\ H(x,1) = \gamma'(x) \end{array}\right\} \forall x \in S^1$

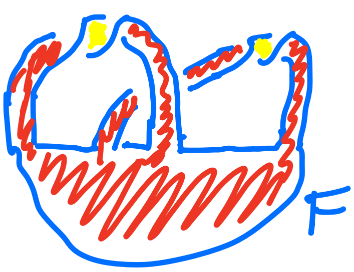
$$H(1,t) = * \in X \quad \forall\, t \in I$$

If we set $\gamma_t(x) = H(x,t)$ $\forall\, x \in S^1$ then $\{\gamma_t\}$ is a path of loops with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$.

Chapter 6 Seifert matrix and signature

Using Seifert's algorithm, one can construct an oriented surface F for any knot diagram K.

One can deform that surface so it is the union of a single 2-disk with $2g$ bands $b_1, \ldots, b_{2g}$ attached.



The surface has one side red and the other is yellow.

Ex: This is a disk-band surface F for the RH trefoil.

In general, one can arrange the bands so they are attached with feet alternating.

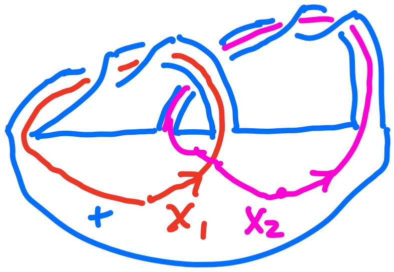
Note: each band must have an even number of half twists.

We can define a $2g \times 2g$ matrix associated to this surface called the <u>Seifert matrix</u>

Let $x_1, x_2, \ldots, x_{2g-1}, x_{2g}$ be cores of the bands, oriented counterclockwise, and let x_i^* be the positive pushoff of x_i using the orientation of the surface.

Define $V = (v_{ij})$ where $v_{ij} = lk(x_i, x_j^*)$
So $lk(x_i, x_j^*)$ counts the places where x_i crosses over x_j^* with sign $+1$ if the crossing is right handed and -1 if it is left handed.

Ex



$$V = \begin{bmatrix} lk(x_1,x_1^*) & lk(x_1,x_2^*) \\ lk(x_2,x_1^*) & lk(x_2,x_2^*) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned}
\Delta_K(t) &= \det(V - tV^T) \\
&= \det\left(\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} - t \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} -1+t & 1 \\ -t & -1+t \end{bmatrix} \right) \\
&= (-1+t)^2 + t = t^2 - 2t + 1 + t \\
&= t^2 - t + 1
\end{aligned}$$

Defn (Knot signature) The signature of the symmetric matrix $A = V + V^T$ is called the <u>knot signature</u>.

Ex $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = A$

$\det A = 3$ $\quad$ $\operatorname{sign} A = -2$ (it has two neg. e.v.s)

$\sigma(3_1) = -2$

<u>Exercise</u> Show that $\sigma(3_1^*) = 2$ for the left-handed trefoil.