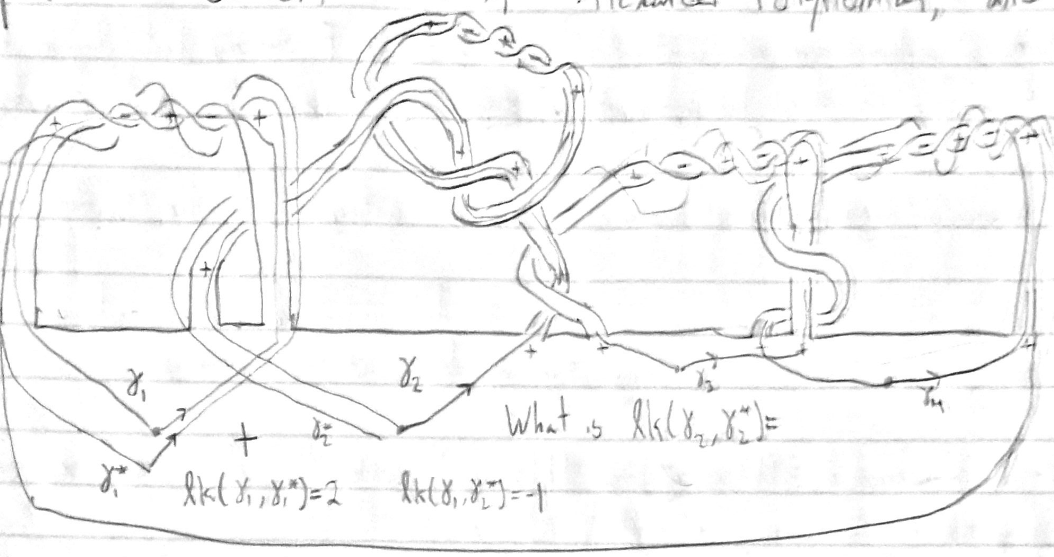
March 15

Chapter 6 Seifert Matrix, Alexander Polynomial, and Knot Signature

Picture

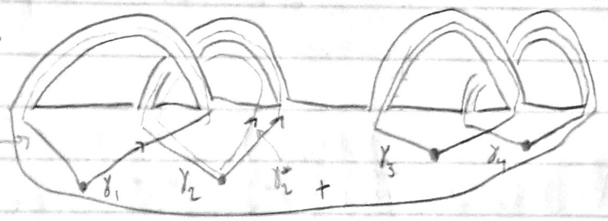
+ = Top
− = Bottom

Ex 1:

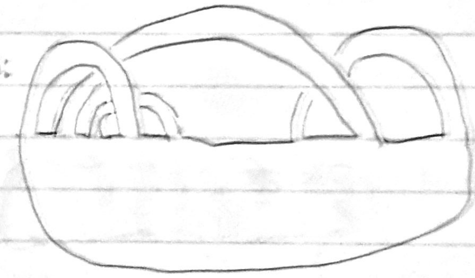


Ex 2:

Cores of bands



Ex 3:



Geometric Techniques, meaning we will use surfaces to study knots.

Recall:

For any knot or link diagram Seiferts construction produces a connected orientated surface whose (orientated) boundary is the knot or link.

We can arrange by isotopy that the surface F is a union of a single disk with bands attatched

The cores of the bands along with arcs in the disk can be used to construct a family of curves on F. We have one curve for each band. Orient the curves arbitrarily The twisting and knotting of the bands reflects properties of the knot and we can record the information using the Siefert matrix.

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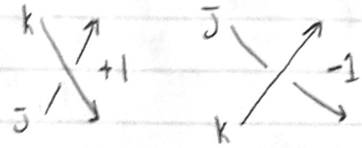
Recall: Linking Number

If $L = K \cup J$ is an orientated link with two components then its linking number can be computed from any regular diagram for L by counting with sign each crossing when K crosses over J. The sign is $+1$ if it is a RH crossing and -1 if its LH.

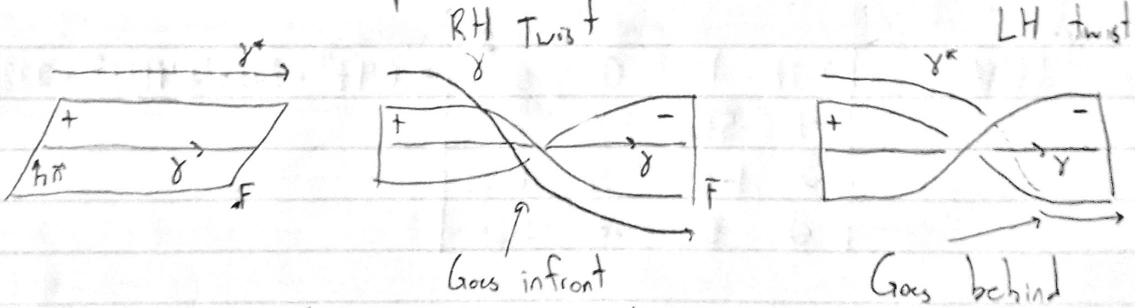


We denote this by $\ell k(K, J)$

Exercise: Show $\ell k(J,K) = \ell k(K,J)$ and $\ell k(J,-K) = -\ell k(J,K) = \ell k(-J,K)$

We will also make use of the fact that the Siefert surface F is orientated, ie it has a "top" side. This is equivalent to choosing a normal vector to F, indicating the positive or top of the surface. A $+$ denotes the top, and a $-$ denotes the bottom

If $\gamma \subset F$ is a curve on F, the positive push off of γ is a parallel curve pushed a small amount in the positive direction



If $\gamma_1, \dots, \gamma_n$ are the bands, the Siefert Matrix is given by

$$V = [v_{ij}] \quad \text{where} \quad v_{ij} = \ell k(\gamma_i, \gamma_j^*)$$

From Ex 2: $\ell k(\gamma_1, \gamma_2^*) = 1$ $\quad \ell k(\gamma_1, \gamma_1^*) = 0$

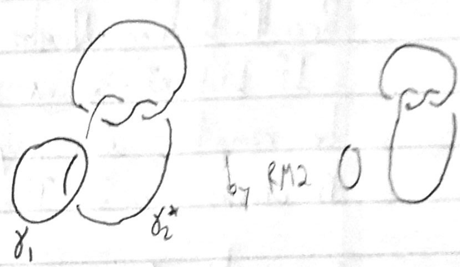
Siefert Matrix

Example

From example 1 we compute the Siefert Matrix

$$V = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$lk(\gamma_2, \gamma_1^*) = 0$ thinks



Theorem:

If K is a knot with Siefert matrix V then its Alexander polynomial is given by $\Delta_K(t) \doteq \det(V - tV^T)$

Remark: 1) $\doteq$ means equal up to $\pm t^k$.

2) Only well defined up to multiplication by $\pm t^k$

3) V^T is the transpose of V

Example

$$\det(V - tV^T) = \begin{vmatrix} 2-2t & 1 & 0 & 0 \\ -t & -5+5t & 1-t & 0 \\ 0 & 1-t & 2-2t & -1+2t \\ 0 & 0 & -2+t & -2+2t \end{vmatrix} = 64t^4 - 272t^3 + 417t^2 - 272t + 64$$

Corollary:

The alexander polynomial of a knot satisfies $\Delta_K(t^{-1}) \doteq \Delta_K(t) = t^k \Delta_K(t)$

Proof

Notice that for any square matrix $\det(V^T) = \det(V)$.

$$\begin{aligned} \Delta_K(t) &\doteq \det(V - tV^T) = \\ &= \det((V - tV^T)^T) \\ &= \det(V^T - tV) \\ &= (-1)^n \det(tV - V^T) \\ &= t^n \det(V - t^{-1}V^T) \\ &= t^n \Delta_K(t^{-1}) \doteq \Delta_K(t^{-1}) \end{aligned}$$

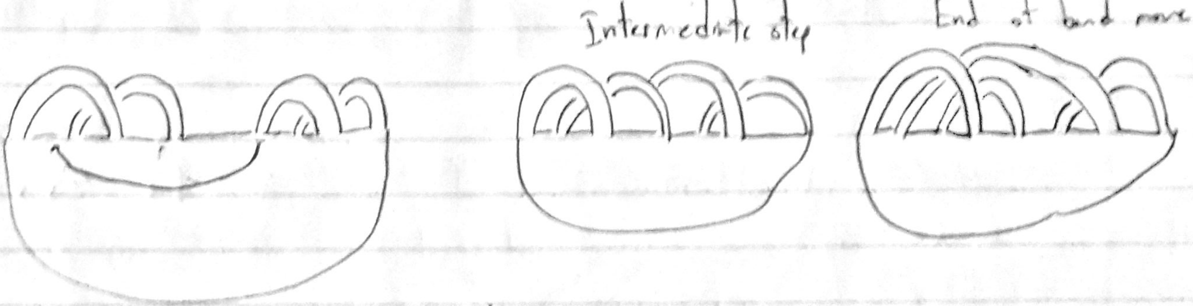
(since n is even $(-1)^n = 1$) $n = \#$ of bands

$n = 2g$ where g is genus

The Siefert surface and matrix is not an invariant of the knot, as they depend on many choices. They can be altered by two moves
i) Band move ii) stabilization.

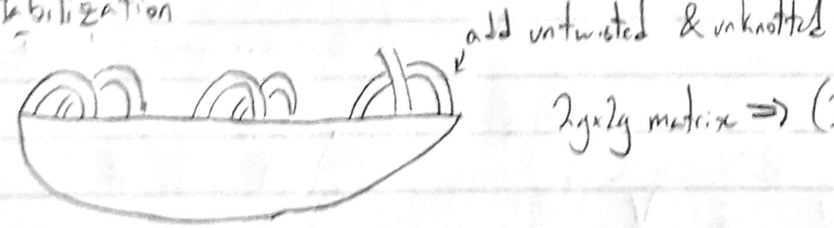
Band Move

A band move is an isotopy of the surface where a foot of one band slides over another band



Algebraically this alters the Siefert Matrix by a row and column operation. ie we add some multiple of row i to row j, and we add the corresponding column i to column j.

Stabilization



$2g \times 2g$ matrix $\Rightarrow (2g+2) \times (2g+2)$ matrix