Splitting the spectral flow and the SU(3) Casson invariant for spliced sums

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We show that the $SU(3)$ Casson invariant for spliced sums along certain torus knots equals 16 times the product of their $SU(2)$ Casson knot invariants. The key step is a splitting formula for $su(n)$ spectral flow for closed 3–manifolds split along a torus.

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1 Introduction

Given knots K_1 and K_2 in homology 3-spheres M_1 and M_2, respectively, the spliced sum of M_1 and M_2 along K_1 and K_2 is the homology 3-sphere obtained by gluing the two knot complements along their boundaries matching the meridian of one knot to the longitude of the other. This operation is a generalization of connected sum; indeed when K_1 and K_2 are trivial knots, the spliced sum of M_1 and M_2 along K_1 and K_2 is none other than the connected sum $M_1 \# M_2$. Casson's invariant $\lambda_{SU(2)}$, which is additive under connected sum, is also additive under the more general operation of spliced sum by Boyer and Nicas [6] and independently Fukuhara and Maruyama [10]. What is remarkable about this is that the Casson invariant of a spliced sum does not depend on the knots K_1 and K_2 along which the splicing is performed.

While the integer-valued $SU(3)$ Casson invariant $\tau_{SU(3)}$ of [3] is not additive under connected sum, by [3, Theorem 4], the difference $\tau_{SU(3)} - 2\lambda^2_{SU(2)}$ is, and a natural question to ask is whether it is also additive under spliced sum. In general, the answer is no and we briefly explain why not. Recall from Saveliev [15] that a Seifert-fibered homology sphere $\Sigma(p,q,r,s)$ can be described as a spliced sum of Brieskorn spheres along the cores of their singular fibers in three different ways: (i) the spliced sum of $\Sigma(p,q,rs)$ and $\Sigma(r,s,pq)$; (ii) the spliced sum of $\Sigma(p,s,qr)$ and $\Sigma(q,r,ps)$; and (iii) the spliced sum of $\Sigma(p,r,qs)$ and $\Sigma(q,s,pr)$. Additivity under splicing would imply that the evaluation of $\tau_{SU(3)} - 2\lambda^2_{SU(2)}$ on all three of these pairs of Brieskorn spheres agree, but the results in [4] provide examples where they do not. This shows that $\tau_{SU(3)} - 2\lambda^2_{SU(2)}$ is not additive under spliced sum.

Thus, it is an interesting problem to understand the behaviour of the $SU(3)$ Casson invariant under spliced sum, and in this paper we focus on the simplest possible case, namely when K_1 and K_2 are torus knots. We verify a conjecture in [2] by identifying the $SU(3)$ Casson invariant of the spliced sum with a multiple of the product of the Casson $SU(2)$ knot invariants in the case K_1 and K_2 are $(2,q_1)$ and $(2,q_2)$ torus knots. Our results combine a detailed analysis of the $SU(3)$ representation varieties of the knot complements with computations of the $su(3)$ spectral flow of the odd signature operator coupled to a path of $SU(3)$ connections. An essential tool developed here is the general splitting formula of Theorem 2.10, which is applied to compute the spectral flow for closed 3–manifolds split along a torus.

We now outline the argument and highlight the special role played by the splitting formula. We assume K_1 and K_2 are knots in S^3 and we denote by X_1 and X_2 their complements and by $M = X_1 \cup_T X_2$ their spliced sum. In Section 3, we give a description of the components of the variety $R(M, SU(3))$ of $SU(3)$ representations of $\pi_1 M$ under certain transversality assumptions on the images of $R(X_i, SU(3))$ in the $SU(3)$ pillowcase $R(T, SU(3))$. In particular, it follows from our description that every component C of $R(M, SU(3))$ with $\dim C > 0$ has $\chi(C) = 0$, and thus by [2, Theorem 7], it follows that only 0–dimensional components contribute to the $SU(3)$ Casson invariant. For instance, this generalizes [2, Theorem 14] and shows that the correction term $\lambda''_{SU(3)}(M)$ for the Casson $SU(3)$ invariant must vanish. Because $\lambda''_{SU(3)}(M) = 0$ and our earlier analysis of the components, we see that

$$\lambda_{SU(3)}(M) = \lambda'_{SU(3)}(M) = \sum_{[A]\in\mathcal{M}^0_{SU(3)}} (-1)^{\mathrm{SF}(\Theta,A)},$$

where $\mathcal{M}^0_{SU(3)}$ denotes the moduli space of isolated, irreducible, flat $SU(3)$ connections on M. The integer-valued invariant $SU(3)$ Casson invariant $\tau_{SU(3)}(M)$ can be analyzed with similar considerations, and in fact it is not difficult to show that $\tau_{SU(3)}(M) = \lambda_{SU(3)}(M)$. In any case, this outlines a straightforward approach for computing the $SU(3)$ Casson invariants for spliced sums.

We carry out these computations in the specific case where M is the spliced sum along two torus knots of type $(2,q_1)$ and $(2,q_2)$. Any representation $\alpha\colon \pi_1(M) \to SU(3)$ determines, by restriction, representations $\alpha_1\colon \pi_1(X_1) \to SU(3)$ and $\alpha_2\colon \pi_1(X_2) \to SU(3)$. We show that the conjugacy class $[\alpha]$ is isolated (and hence contributes nontrivially to the $SU(3)$ Casson invariant of M) only when α is irreducible and both α_1 and α_2 are reducible. By conjugating, we can arrange that α_1 is an $S(U(2)\times U(1))$

representation and that α_2 is an $S(U(1) \times U(2))$ representation. To complete the computation, we just need to enumerate all such representations and determine the $su(3)$ spectral flow from the trivial connection Θ to the flat connection A on M corresponding to α.

Since the $S(U(2) \times U(1))$ representation variety of a torus knot is connected, there is a path $A_{1,t}$ of flat $S(U(2) \times U(1))$ connections on X_1 connecting $\Theta|_{X_1}$ to $A|_{X_1}$, and likewise a path $A_{2,t}$ of flat $S(U(1) \times U(2))$ connections on X_2 connecting $\Theta|_{X_2}$ to $A|_{X_2}$. Moreover, these paths can be chosen to satisfy the hypotheses in Theorem 2.10. The splitting theorem then describes the spectral flow on the spliced sum M as a sum of the spectral flows of the paths $A_{1,t}$ and $A_{2,t}$ of *flat* connections on knot complements X_1 and X_2, the spectral flow of a closed path of $SU(3)$ connections on the solid torus, and some finite dimensional Maslov triple indices. Each of these terms can be computed by direct analysis, and from this we deduce our main application, which identifies the $SU(3)$ Casson invariant of the spliced sum with a multiple of the product of the $SU(2)$ Casson knot invariants for spliced sums along certain torus knots. The following is a restatement of Theorem 7.6, our main result.

Theorem Suppose K_1 and K_2 are torus knots of type $(2, q_1)$ and $(2, q_2)$, respectively, and M is their spliced sum. Then

$$\lambda_{SU(3)}(M) = 16\, \lambda'_{SU(2)}(K_1)\ \lambda'_{SU(2)}(K_2),$$

where $\lambda'_{SU(2)}(K)$ is the $SU(2)$ Casson knot invariant normalized to be 1 for the trefoil.

Remark As noted above, this result also computes the integer-valued $SU(3)$ Casson invariant $\tau_{SU(3)}(M)$ of [3]. While the results of [5] and [4] show that neither the $SU(3)$ Casson invariant has finite-type, the above theorem shows that the behavior of $\lambda_{SU(3)}$ and $\tau_{SU(3)}$ under splicing is very similar to that of the finite-type invariant of degree three. Note that additivity of the Casson $SU(2)$ invariant under spliced sum [6, 10], implies that $\lambda_{SU(2)}(M) = 0$ for any 3–manifold M obtained as the spliced sum along two knots in S^3.

Here is a brief synopsis of the rest of the paper. In Section 2 we present the splitting theorem in the general setting. Section 3 contains some general results about $SU(3)$ representations of spliced sums, and Section 4 and Section 5 give descriptions of the reducible and irreducible $SU(3)$ representations of torus knots. Section 6 contains cohomology calculations, and Section 7 presents the main application to computing the $SU(3)$ Casson invariant for spliced sums along torus knots.

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2 A splitting formula for *su(n)* spectral flow

The $SU(3)$ Casson invariant for homology 3-spheres is defined in [3] by counting gauge orbits of irreducible (perturbed) flat $SU(3)$ connections with sign given by the $su(3)$ spectral flow. In the case of a 3–manifold split along a surface, a useful tool for performing computations of the spectral flow is provided by splitting the spectral flow along the manifold decomposition. Existing splitting formulas treat mainly the $SU(2)$ case and do not readily apply to our situation, so in this section our goal is to develop a suitably general splitting formula for 3–manifolds split along a torus. Our results here are the natural $su(n)$ generalizations of the results established in [11] for $su(2)$ spectral flow, and the arguments that are routine extensions of those given in [11] will only be outlined.

When working on manifolds with boundary, it is essential to have a family or at least a path of "nice" boundary conditions associated to the restriction of A_t to the boundary (see Atiyah–Patodi–Singer [1]). For example, given a path of Atiyah–Patodi–Singer boundary conditions, we could derive a splitting formula for arbitrary splitting surfaces, however, in general we cannot find a path of Atiyah–Patodi–Singer boundary conditions for a given path of connections which is continous in the gap topology. We note that before choosing boundary conditions we may assume any path between flat connections to stop a finite number of times and to be flat on the boundary torus, because the spectral flow is homotopy invariant. Therefore, in Section 2.2, we describe an explicit family of boundary conditions together with a family of flat connections on the boundary torus which is suitable for all the spectral flow computations we have in mind.

This section assumes some level of familiarity with the background material on spectral flow, Maslov index, and their relationship. Readers interested in learning more about these aspects are referred to Cappell-Lee–Miller [7] and Nicolaescu [14].

2.1 Preliminaries

In order to describe the family of Atiyah–Patodi–Singer boundary conditions and formulate the splitting formula, we recall the basic setup and review the concepts and notation that will be used throughout this section.

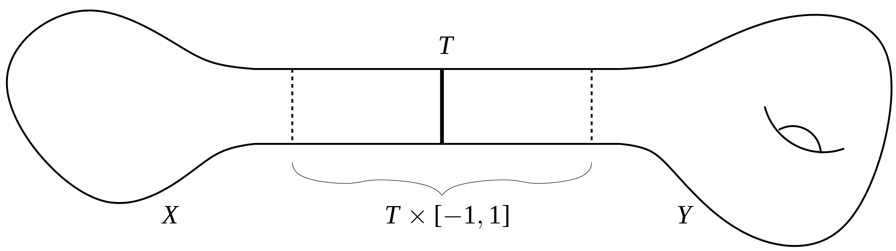

Figure 1: The collar around T

For the splitting formula we will assume the following:

(1) The orientation of the torus $T = S^1 \times S^1 = \{(e^{im}, e^{i\ell}) \mid m, \ell \in [0, 2\pi)\}$ is determined by $dm \wedge d\ell \in \Omega^2(T)$. We regard T with its product metric from the standard metric on S^1, and note that the fundamental group $\pi_1(T)$ is the free abelian group generated by the meridian $\mu = \{(e^{im}, 1)\}$ and longitude $\lambda = \{(1, e^{i\ell})\}$.

(2) The 3–manifolds X and Y have boundary T and are oriented so that $\partial X = T = -\partial Y$. We place metrics on X and Y such that collars of ∂X and ∂Y are isometric to $[-1, 0] \times T$ and $[0, 1] \times T$, respectively.

(3) Consider the 3–manifold $M = X \cup_T Y$ with the orientation and metric induced by the orientation and metric on X and Y. See Figure 1.

(4) Fix a principal bundle with structure group $SU(n)$ over M and consider its trivialization.

For an $SU(n)$ connection $A \in \Omega^1(M; su(n))$, the *odd signature operator twisted by A* is defined to be

$$\begin{aligned} D_A \colon \Omega^{0+1}(M; su(n)) &\to \Omega^{0+1}(M; su(n)) \\ (\alpha, \beta) &\mapsto (d_A^* \beta, \star d_A \beta + d_A \alpha), \end{aligned}$$

where $\Omega^{0+1}(M; su(n)) = \Omega^0(M) \otimes su(n) \,\oplus\, \Omega^1(M) \otimes su(n)$ and $\star$ denotes the Hodge star operator on the 3–manifold M. For an $SU(n)$ connection $a \in \Omega^1(T; su(n))$, the *de*

Rham operator twisted by a is defined to be

$$\begin{aligned} S_a\colon\ \Omega^{0+1+2}(T;su(n)) &\to \Omega^{0+1+2}(T;su(n)) \\ (\alpha,\beta,\gamma) &\mapsto (*d_a\beta, -*d_a\alpha - d_a*\gamma, d_a*\beta), \end{aligned}$$

where $*$ denotes the Hodge star operator on the 2–manifold T.

If a is flat, then the Laplacian twisted by a is given by $\Delta_a = S_a^2$, which is an operator

$$\Delta_a\colon\ \Omega^{0+1+2}(T;su(n)) \to \Omega^{0+1+2}(T;su(n)).$$

We equip $L^2(\Omega^{0+1+2}(T;su(n)))$ with an almost complex structure J by setting

$$J(\alpha,\beta,\gamma) := (-*\gamma, *\beta, *\alpha). \tag{2–1}$$

Let A be a connection on M, which is in cylindrical form in a collar of T, that is $A = i_u^* a$, where $i_u\colon\ T \hookrightarrow [-1,1]\times T$ is the inclusion at $u\in[-1,1]$ and a is a connection on T. We define the following function

$$\begin{aligned} r\colon\ \Omega^{0+1}([-1,1]\times T;su(n)) &\to \Omega^{0+1+2}(T;su(n)) \\ (\sigma,\tau) &\mapsto \left(i_0^*(\sigma), i_0^*(\tau), *i_0^*\left(\tau\lrcorner\tfrac{\partial}{\partial u}\right)\right), \end{aligned}$$

where $\tau\lrcorner\frac{\partial}{\partial u}$ denotes contraction of τ with $\frac{\partial}{\partial u}$. This also gives us a restriction map of $\Omega^{0+1}(X;su(n))$ and $\Omega^{0+1}(Y;su(n))$ to $\Omega^{0+1+2}(T;su(n))$. The *Cauchy data spaces* of $D_A|_X$ and $D_A|_Y$ are

$$\Lambda_{X,A} := \overline{r(\operatorname{Ker} D_A|_X)}^{L^2} \quad\text{and}\quad \Lambda_{Y,A} := \overline{r(\operatorname{Ker} D_A|_Y)}^{L^2}, \quad\text{respectively,}$$

with the corresponding *limiting values of extended L^2-solutions*

$$\mathcal{L}_{X,A} := \operatorname{proj}_{\operatorname{Ker} S_a}(\Lambda_{X,A}\cap(P^-\cup\operatorname{Ker} S_a))$$

and

$$\mathcal{L}_{Y,A} := \operatorname{proj}_{\operatorname{Ker} S_a}((P^+\cup\operatorname{Ker} S_a)\cap\Lambda_{Y,A}).$$

We may attach a collar to X and Y and define

$$\Lambda^R_{X,A} := \overline{r(\operatorname{Ker} D_A|_{X\cup[0,R]})}^{L^2} \quad\text{and}\quad \Lambda^R_{Y,A} := \overline{r(\operatorname{Ker} D_A|_{Y\cup[-R,0]})}^{L^2},$$

as well as

$$\Lambda^\infty_{X,A} := \overline{r(\operatorname{Ker} D_A|_{X\cup[0,\infty)})}^{L^2} \quad\text{and}\quad \Lambda^\infty_{Y,A} := \overline{r(\operatorname{Ker} D_A|_{Y\cup(-\infty,0]})}^{L^2}.$$

2.2 A family of Atiyah–Patodi–Singer boundary conditions

In this subsection, we construct a family of flat connections $a_{\tilde\varrho}$ on the 2–torus T and introduce boundary conditions $\mathscr{P}_{\tilde\varrho}$ parametrized by $\tilde\varrho\in\widetilde\Lambda$ with the property that, for 3–manifolds X with $\partial X = T$, given any connection A on X whose restriction $A|_T$ is flat, then A is gauge equivalent to some A' with $A'|_T = a_{\tilde{\varrho}}$ for some $\tilde{\varrho} \in \widetilde{\Lambda}$ and $\mathscr{P}_{\tilde{\varrho}}$ is an Atiyah–Patodi–Singer boundary condition for the twisted odd signature operator $D_{A'}$, ie $\mathscr{P}_{\tilde{\varrho}}$ contains all eigenvectors of the tangential operator $S_{a_{\tilde{\varrho}}}$ with sufficiently large eigenvalues. Furthermore, we describe a natural topology on $\widetilde{\Lambda}$ in which it continuously parametrizes the family of boundary conditions and flat connections.

Let $R(T, SU(n))$ be the representation variety of T, namely the space of conjugacy classes of representations $\varphi\colon \pi_1(T) \to SU(n)$. By Donaldson and Kronheimer [9, Proposition 2.2.3], the holonomy map gives a homeomorphism from the moduli space $\mathscr{M}_T$ of flat $SU(n)$ connections over T to the representation variety $R(T, SU(n))$.

Let $\Lambda := \{\alpha = (\alpha_1 \ldots, \alpha_n) \in \mathbb{R}^n \mid \alpha_1 + \cdots + \alpha_n = 0\}$, which is isomorphic to $\mathbb{R}^{n-1}$ via the standard projection onto the first $n-1$ coordinates. For $\alpha \in \Lambda$, set

$$\operatorname{diag}(\alpha) = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}.$$

Definition 2.1 For $\alpha, \beta \in \Lambda$, let $a_{\alpha,\beta} := -i\operatorname{diag}(\alpha)\, dm - i\operatorname{diag}(\beta)\, d\ell$. We substitute an index $a_{\alpha,\beta}$ by (α, β), for example $S_{\alpha,\beta} = S_{a_{\alpha,\beta}}$, $\Delta_{\alpha,\beta} = \Delta_{a_{\alpha,\beta}}$.

Notice that $a_{\alpha,\beta}$ is a flat connection on T with holonomy $\operatorname{hol}(a_{\alpha,\beta})$ equal to the representation $\varphi_{\alpha,\beta}\colon \pi_1(T) \to SU(n)$ given by $\varphi_{\alpha,\beta}(\mu) = \exp(2\pi i \operatorname{diag}(\alpha))$ and $\varphi_{\alpha,\beta}(\lambda) = \exp(2\pi i \operatorname{diag}(\beta))$. The map $(\alpha, \beta) \mapsto a_{\alpha,\beta}$ defines a smooth family of flat connections parameterized by Λ^2, and the map $(\alpha, \beta) \mapsto [\varphi_{\alpha,\beta}]$ gives a branched cover $\Lambda^2 \to R(T, SU(n))$.

Under the action of the standard maximal torus $T^{n-1} \subset SU(n)$, the Lie algebra decomposes as $su(n) = U_n \oplus W_n$ into diagonal and off-diagonal parts. The torus acts trivially on the diagonal part $U_n \cong \mathbb{R}^{n-1}$ and nontrivially on the off-diagonal part W_n, which further decomposes as $W_n = \bigoplus_{i<j} C^{ij}$, where

$$C^{ij} := \{a \in su(n) \mid a_{kl} = 0 \text{ for } \{k,l\} \neq \{i,j\}\} \cong \mathbb{C}.$$

Moreover, $S_{\alpha,\beta}$ and $\Delta_{\alpha,\beta}$ preserve the induced splitting of $\Omega^{0+1+2}(T; su(n))$. Therefore, the detailed analysis of our boundary conditions can be done for $U_n \cong \mathbb{R}^{n-1}$ and $W_n = \bigoplus_{i<j} C^{ij}$ by effectively reducing them to the computations in [11]. Notice that $W_n \cong \mathbb{C}^{\binom{n}{2}}$.

For $i < j$, we define subspaces

$$Q^{ij}_{\alpha,\beta} = \operatorname{span}_{\mathbb{C}}(\phi^{ij}) \subset \Omega^0(T; W_n), \tag{2–2}$$

where $\phi^{ij} = (\phi^{ij}_{kl}) \in \Omega^0(T; W_n)$ is given by

$$\phi^{ij}_{kl}(m,\ell) = \begin{cases} e^{i((\alpha_i-\alpha_j)m+(\beta_i-\beta_j)\ell)} & \text{if } (k,l)=(i,j), \\ -e^{i((\alpha_j-\alpha_i)m+(\beta_j-\beta_i)\ell)} & \text{if } (k,l)=(j,i), \\ 0 & \text{otherwise.} \end{cases} \tag{2–3}$$

We set

$$Q_{\alpha,\beta} = \bigoplus_{i<j} Q^{ij}_{\alpha,\beta}. \tag{2–4}$$

For a proof of the next result, see [11, Proposition 3.1.2].

Proposition 2.2 *We have for the harmonic forms of $\Delta_{\alpha,\beta}$ on the torus:*

$$\mathscr{H}^{0+1+2}_{\alpha,\beta}(T; su(n)) = \mathscr{H}^{0+1+2}_{\alpha,\beta}(T; U_n) \oplus \mathscr{H}^{0+1+2}_{\alpha,\beta}(T; W_n).$$

In the first case, we have trivially that

$$\mathscr{H}^{i}_{\alpha,\beta}(T; U_n) = \begin{cases} U_n, & \text{if } i=0, \\ U_n\, dm \oplus U_n\, d\ell, & \text{if } i=1, \text{ and} \\ U_n\, dm \wedge d\ell, & \text{if } i=2. \end{cases}$$

In the second case, we have

$$\mathscr{H}^{0+1+2}_{\alpha,\beta}(T; W_n) = \bigoplus_{i<j} \mathscr{H}^{0+1+2}_{\alpha,\beta}(T; C^{ij}),$$

with

$$\mathscr{H}^{0}_{\alpha,\beta}(T; C^{ij}) = \begin{cases} Q^{ij}_{\alpha,\beta} & \text{if } (\alpha_i-\alpha_j, \beta_i-\beta_j) \in \mathbb{Z}^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathscr{H}^{1}_{\alpha,\beta}(T; C^{ij}) = \begin{cases} Q^{ij}_{\alpha,\beta}\, dm \oplus Q^{ij}_{\alpha,\beta}\, d\ell & \text{if } (\alpha_i-\alpha_j, \beta_i-\beta_j) \in \mathbb{Z}^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathscr{H}^{2}_{\alpha,\beta}(T; C^{ij}) = \begin{cases} Q^{ij}_{\alpha,\beta}\, dm \wedge d\ell & \text{if } (\alpha_i-\alpha_j, \beta_i-\beta_j) \in \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

Let a be an $SU(n)$ connection on T and $E_{a,\nu}$ denote the ν–eigenspace of S_a. For $\nu > 0$, we set

$$\begin{aligned} P^+_{a,\nu} &:= \operatorname{span}_{L^2} \{\psi \mid S_a\psi = \mu\psi \text{ for } \mu > \nu\}, \\ P^-_{a,\nu} &:= \operatorname{span}_{L^2} \{\psi \mid S_a\psi = \mu\psi \text{ for } \mu < -\nu\}, \\ E^+_{a,\nu} &:= \operatorname{span}_{L^2} \{\psi \mid S_a\psi = \mu\psi \text{ for } 0 < \mu \leq \nu\}, \end{aligned}$$

and

$$E^-_{a,\nu} := \operatorname{span}_{L^2} \{\psi \mid S_a\psi = \mu\psi \text{ for } -\nu \leq \mu < 0\}.$$

Notice that

$$P^{\pm}_{a,\nu} := \overline{\bigoplus_{\pm\mu>\nu} E_{a,\mu}}^{L^2} \quad \text{and} \quad E^{\pm}_{a,\nu} := \bigoplus_{0<\pm\mu\leq\nu} E_{a,\mu}.$$

If $\nu = 0$, we write $P^{\pm}_{a}$ in place of $P^{\pm}_{a,0}$, and if $\alpha,\beta \in \Lambda$, we write $P^{\pm}_{\alpha,\beta}$ in place of $P^{\pm}_{a_{\alpha,\beta}}$. Define $P^{ij\pm}_{\alpha,\beta} := P^{\pm}_{\alpha,\beta} \cap L^2(\Omega^{0+1+2}(T;C^{ij}))$. Observe that the space of twisted harmonic forms $\mathscr{H}^{0+1+2}_a(T;su(n))$ in $L^2(\Omega^{0+1+2}(T,su(n)))$ is equal to $\operatorname{Ker} S_a$. By the spectral theorem for self-adjoint elliptic operators we have

$$L^2(\Omega^{0+1+2}(T,su(n))) = P^{+}_{a} \oplus \operatorname{Ker} S_a \oplus P^{-}_{a}.$$

Just as in [11, Proposition 3.2.3], we get a decomposition of $L^2(\Omega^{0+1+2}(T,su(n)))$ into eigenspaces of $\Delta_{\alpha,\beta}$ respecting the decompositions $su(n) = U_n \oplus W_n$ and $W_n = \bigoplus_{i<j} C^{ij}$. Further note that the decomposition of $L^2(\Omega^{0+1+2}(T,U_n))$ is independent of (α,β) and the decomposition of $L^2(\Omega^{0+1+2}(T,C^{ij}))$ depends only on $(\alpha_i-\alpha_j,\beta_i-\beta_j) \in \mathbb{R}^2$. The dimension of $\operatorname{Ker} S_{\alpha,\beta}$ jumps whenever $(\alpha_i-\alpha_j,\beta_i-\beta_j)$ lies in the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ for some $i<j$. We set

$$\begin{aligned}
\mathcal{Z}_{ij} &:= \{(\alpha,\beta) \in \Lambda^2 \mid (\alpha_i-\alpha_j,\beta_i-\beta_j) \in \mathbb{Z}^2\},\\
\mathcal{Z} &:= \bigcup_{i<j} \mathcal{Z}_{ij}.
\end{aligned}$$

Remark As spectral flow on a closed manifold is an invariant of homotopy rel endpoints, for the purpose of the spectral flow calculations in this paper, we can always assume that if a path $(\alpha(t),\beta(t))$ hits $\mathcal{Z}_{ij}$ when $t=t_0$, then it approaches $\mathcal{Z}_{ij}$ in such a way that

$$\frac{\beta_j(t)-\beta_i(t)}{\alpha_j(t)-\alpha_i(t)} = \tan\theta_{ij}, \quad \theta_{ij} \in S^1,$$

is constant on some interval $t \in (t_0-\epsilon,t_0)$, and similarly for when it leaves $\mathcal{Z}_{ij}$. For such a path, the kernel of $S_{\alpha(t),\beta(t)}$ converges as $t \to t_0$ and the kernel at t_0 equals this limit plus an additional subspace determined by θ_{ij}.

We take an alternative approach and shall introduce a parameter space $\widetilde{\Lambda^2}$ with topology so that every continuous path in $\widetilde{\Lambda^2}$ is "sufficiently nice" in an appropriate sense. This viewpoint has a conceptual advantage and provides additional flexibility, because the spectral flow along a path of connections on a manifold with torus boundary is homotopy invariant rel endpoints, as long as its restriction respects a certain family of boundary conditions together with flat connections parametrized by $\widetilde{\Lambda^2}$.

By explicitly computing some sufficiently nice path of eigenfunctions with nonzero eigenvalue, which vanish in the limit, we can see that the additional eigenspace in the kernel of the tangential operator only depends on the direction in which $(\alpha_i - \alpha_j, \beta_i - \beta_j)$ approaches $\mathbb{Z}^2$. We will make this precise. For $i < j$, we denote this angle by $\theta_{ij} \in S^1$, and we introduce the parameter space

$$\widetilde{\Lambda^2} := \Lambda^2 \times (S^1)^{\binom{n}{2}} / \sim,$$

where the equivalence relation collapses the (ij) circle away from $\mathcal{Z}_{ij}$, ie for $\theta = (\theta_{ij})_{i<j} \in (S^1)^{\binom{n}{2}}$, we have

$$(\alpha, \beta, \theta) \sim (\alpha, \beta, \theta') \text{ provided } \theta_{ij} = \theta'_{ij} \text{ for all } i < j \text{ with } (\alpha, \beta) \in \mathcal{Z}_{ij}.$$

We put a topology on $\widetilde{\Lambda^2}$ as follows. Given $(\alpha, \beta) \in \Lambda^2$ and $i < j$, set $\alpha_{ij} = \alpha_i - \alpha_j$ and $\beta_{ij} = \beta_i - \beta_j$. Then $(\alpha_{ij}, \beta_{ij})_{i<j} \in (\mathbb{R}^2)^{\binom{n}{2}}$. Set $\Omega^2 = (\mathbb{R}^2)^{\binom{n}{2}}$ for notational convenience, and notice that the map $\Lambda^2 \to \Omega^2$ given by $(\alpha, \beta) \mapsto (\alpha_{ij}, \beta_{ij})_{i<j}$ is an embedding. As before, define

$$\widetilde{\Omega^2} = \Omega^2 \times (S^1)^{\binom{n}{2}} / \sim,$$

where the equivalence relation collapses the (ij) circle for $(\alpha_{ij}, \beta_{ij}) \notin \mathbb{Z}^2$. Just as on p. 2275 of [11], there is a bijective map from $\widetilde{\Omega^2}$ to $(\dot{\mathbb{R}}^2)^{\binom{n}{2}}$, where $\dot{\mathbb{R}}^2$ is the result of removing open disks of radius $1/4$ around each integer lattice point in $\mathbb{R}^2$, and we put a topology on $\widetilde{\Omega^2}$ that makes this map a homeomorphism. The embedding $\Lambda^2 \times (S^1)^{\binom{n}{2}} \to \Omega^2 \times (S^1)^{\binom{n}{2}}$ descends to an injective map $\widetilde{\Lambda^2} \to \widetilde{\Omega^2}$, and in this way $\widetilde{\Lambda^2}$ inherits the pullback topology from $\widetilde{\Omega^2}$.

The next result is analogous to [11, Theorem 3.2.2]. Before stating it, we define families $K^{\pm}_{(\alpha,\beta,\theta)} = \bigoplus_{i<j} K^{ij\pm}_{(\alpha,\beta,\theta)}$ of subspaces of $\mathscr{H}^{0+1+2}_{(\alpha,\beta)}(T; su(n))$ parameterized by $\widetilde{\Lambda^2}$ by setting, for each $i < j$,

$$K^{ij\pm}_{(\alpha,\beta,\theta)} = \begin{cases} \operatorname{span}_{\mathbb{C}}\{\psi_1^{ij\pm}, \psi_2^{ij\pm}\} & \text{if } (\alpha, \beta) \in \mathcal{Z}_{ij}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \psi_1^{ij\pm} &= \phi^{ij} \mp \phi^{ij}(i \operatorname{Im}\theta_{ij}\, dm - i \operatorname{Re}\theta_{ij}\, d\ell), \\ \psi_2^{ij\pm} &= \phi^{ij}\, dm \wedge d\ell \pm \phi^{ij}(i \operatorname{Re}\theta_{ij}\, dm + i \operatorname{Im}\theta_{ij}\, d\ell), \end{aligned}$$

and $\phi^{ij} \in \Omega^0(T; su(n))$ is the function given by equation (2–3).

Theorem 2.3

(1) *The maps* $P^\pm\colon\ \Lambda^2\setminus\mathcal{Z}\to\{\textit{closed subspaces of } L^2(\Omega^{0+1+2}(T,su(n)))\}$ *are continuous.*

(2) *If* $(\alpha(t),\beta(t))\in\Lambda^2$, $t\in[0,\varepsilon)$ *is a smooth path with* $(\alpha(t),\beta(t))\notin\mathcal{Z}_{ij}$ *for* $t\in(0,\varepsilon)$ *such that* $\frac{d}{dt}\big|_{t=0}\big(\alpha_{ij}(t)+i\beta_{ij}(t)\big)\neq 0$, *we set*

$$\theta_{ij}=\frac{\alpha'_{ij}(0)+i\beta'_{ij}(0)}{\|\alpha'_{ij}(0)+i\beta'_{ij}(0)\|}.$$

Then

$$\lim_{t\to0^+}P^{ij+}_{(\alpha(t),\beta(t))}=K^{ij+}_{(\alpha,\beta,\theta)}\oplus P^{ij+}_{(\alpha,\beta)}\quad\textit{and}\quad\lim_{t\to0^+}P^{ij-}_{(\alpha(t),\beta(t))}=K^{ij-}_{(\alpha,\beta,\theta)}\oplus P^{ij-}_{(\alpha,\beta)}.$$

(3) *Extend* $P^\pm$ *to* $\widetilde{\Lambda^2}$ *by setting* $P^\pm_{(\alpha,\beta,\theta)}=P^\pm_{(\alpha,\beta)}$. *Then*

$$P^\pm\oplus K^\pm\colon\ \widetilde{\Lambda^2}\to\{\textit{closed subspaces of } L^2(\Omega^{0+1+2}(T,su(n)))\}$$

are continuous.

□

Then, we can define a continuous family of boundary conditions parametrized by $\widetilde{\Lambda^2}$ (cf [11, Definition 3.2.4]).

Definition 2.4 Define a family $\mathscr{P}^\pm$ of subspaces of $L^2(\Omega^{0+1+2}(T,su(n)))$ continuously parametrized by $\tilde{\varrho}\in\widetilde{\Lambda^2}$ as

$$\mathscr{P}^\pm_{\tilde{\varrho}}:=P^\pm_{\tilde{\varrho}}\oplus\hat{\mathscr{L}}^\pm\oplus K^\pm_{\tilde{\varrho}},$$

where

$$\hat{\mathscr{L}}^-:=U\oplus U\,d\ell\quad\text{and}\quad\hat{\mathscr{L}}^+:=J\hat{\mathscr{L}}^-$$

and J is given in (2–1). The space $\hat{\mathscr{L}}^\pm$ can be chosen arbitrarily–the proof of the splitting formula does not make use of it–but the above choice makes computations for our application easier.

If $L_{1,t}$, $L_{2,t}$ and $L_{3,t}$, $t\in[0,1]$ are paths of Lagrangian subspaces in a symplectic Hilbert space with almost complex structure J, such that $(JL_{i,t},L_{j,t})$ is a Fredholm pair for all $i,j=1,2,3$, $t\in[0,1]$, then we can define a Maslov triple index τ_μ by translating [12, Definition 6.8] by Kirk and Lesch into the language of Lagrangian subspaces. By the proof of [12, Lemma 6.10], we see that τ_μ is determined by $\tau_\mu(L,L,L)=0$ and

$$\tau_\mu(L_{1,1},L_{2,1},L_{3,1})-\tau_\mu(L_{1,0},L_{2,0},L_{3,0})=\mathrm{Mas}(JL_1,L_2)+\mathrm{Mas}(JL_2,L_3)-\mathrm{Mas}(JL_1,L_3).$$

Some easy and useful properties are summarized in the following.

Lemma 2.5 *Let L_1, L_2, and L_3 be pairwise Fredholm Lagrangians in a Hilbert space H. Then*

- $\tau_\mu(L_1, L_1, L_2) = \tau_\mu(L_1, L_2, L_2) = 0$,
- $\tau_\mu(L_1, L_2, L_1) = \dim(JL_1 \cap L_2)$, *and*
- $\tau_\mu(L_1, L_2, L_3) = \dim(JL_2 \cap L_3) - \tau_\mu(L_1, L_3, L_2)$.

2.3 Derivation of the *su(n)* splitting formula

In this subsection we develop a splitting formula which expresses the $su(n)$ spectral flow of the odd signature operator between flat connections on a closed 3–manifold $M = X \cup_T Y$ split along a torus T in terms of spectral flows on X and Y with the Atiyah–Patodi–Singer boundary conditions from Section 2.2.

Theorem 2.6 *Let $M = X \cup_T Y$ be a closed 3–manifold split along the torus T. Let A_t be a path of $SU(n)$ connections on M with the following properties:*

(1) *A_t is in cylindrical form and flat in a collar of T.*

(2) *A_t restricts to the path $a_{\varrho(t)}$ on T for some path $\tilde{\varrho}$ in $\widetilde{\Lambda^2}$ with $\pi \circ \tilde{\varrho} = \varrho$, where $\pi \colon \widetilde{\Lambda^2} \to \Lambda^2$ is the obvious projection, and*

(3) *A_0 and A_1 are flat on M.*

Then we have the splitting formula:

$$\begin{aligned}\mathrm{SF}(A_t) = {}& \mathrm{SF}(A_t|_X; \mathscr{P}^+_{\tilde{\varrho}(t)}) + \mathrm{SF}(A_t|_Y; \mathscr{P}^-_{\tilde{\varrho}(t)}) + \tau_\mu(J\mathscr{L}_{X,\varrho(0)}, K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,\varrho(0)}) \\ & - \tau_\mu(J\mathscr{L}_{X,\varrho(1)}, K^+_{\tilde{\varrho}(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,\varrho(1)}).\end{aligned}$$

Proof The proof is very similar to [11, section 4.4]. Recall from Nicolaescu [14, Definition 4.8] that the non-negative numbers $\min\{\nu \in \mathbb{R} \mid \Lambda_{X,A} \cap P^+_{a,\nu} = 0\}$ and $\min\{\nu \in \mathbb{R} \mid P^-_{a,\nu} \cap \Lambda_{Y,A} = 0\}$ are called the non-resonance levels of $D_A|_X$ and $D_A|_Y$ respectively. Let ν be the maximum of the non-resonance levels of $D_{A_0}|_X$, $D_{A_1}|_X$, $D_{A_0}|_Y$ and $D_{A_1}|_Y$. For $\varepsilon = 0, 1$, we use $E^\pm_{\varepsilon,\nu}$ for the spaces $E^\pm_{a_{\varrho(\varepsilon)},\nu}$ and set

$$H_{\varepsilon,\nu} := E^+_{\varepsilon,\nu} \oplus \operatorname{Ker} S_{\varrho(\varepsilon)} \oplus E^-_{\varepsilon,\nu}.$$

Using the notation from above $\Lambda_{X,t} := \Lambda_{X,A_t}$ and $\Lambda_{Y,t} := \Lambda_{Y,A_t}$:

(1) Fix some path $L_{X,\varepsilon,t}$, $\varepsilon = 0, 1$, of Lagrangians in $H_{\varepsilon,\nu}$ from $\Lambda^\infty_{X,\varepsilon} \cap H_{\varepsilon,\nu}$ to $\mathscr{P}^-_{\tilde{\varrho}(\varepsilon)} \cap H_{\varepsilon,\nu}$, and

(2) fix some path $L_{Y,\varepsilon,t}$, $\varepsilon = 0,1$, of Lagrangians in $H_{\varepsilon,\nu}$ from $\Lambda^\infty_{Y,\varepsilon} \cap H_{\varepsilon,\nu}$ to $\mathscr{P}^+_{\tilde{\varrho}(\varepsilon)} \cap H_{\varepsilon,\nu}$.

Consider the Maslov index of the path $(\Lambda_{X,t}, \Lambda_{Y,t})$. Then by the results of Daniel [8, Theorem 4.3] (see also Nicolaescu [14, Theorem 3.14]), we have that $\mathrm{SF}(A_t) = \mathrm{Mas}(\Lambda_{X,t}, \Lambda_{Y,t})$, as well as the relative version $\mathrm{SF}(A_t|_X; \mathscr{P}^+_{\tilde{\varrho}(t)}) = \mathrm{Mas}(\Lambda_{X,t}, \mathscr{P}^+_{\tilde{\varrho}(t)})$ and $\mathrm{SF}(A_t|_Y; \mathscr{P}^-_{\tilde{\varrho}(t)}) = \mathrm{Mas}(\mathscr{P}^-_{\tilde{\varrho}(t)}, \Lambda_{Y,t})$ by Kirk and Lesch [12, Theorem 7.5]. We can homotope the path $(\Lambda_{X,t}, \Lambda_{Y,t})$ to the concatenation of paths $(\mathscr{M}_i, \mathscr{N}_i)$, $i = 1, \ldots, 11$ given in Table 1 without changing the Maslov index.

Observe first, that the Maslov index of each of the pairs $(\mathscr{M}_i, \mathscr{N}_i)$, $i = 1, 4, 6, 8, 11$ is zero (see [12, Lemma 8.10]).

Furthermore we can apply [12, Theorem 8.5], where $W_X \subset dE^-_{0,\nu} \subset E^+_{0,\nu}$ for $D_{A_0}|_X$ and $W_Y \subset dE^+_{0,\nu} \subset E^-_{0,\nu}$ for $D_{A_0}|_Y$ are as in the theorem, and $\perp$ denotes the orthogonal complement in $dE^-_{0,\nu}$ and $dE^+_{0,\nu}$ respectively, to get

$$\begin{aligned}
&\mathrm{Mas}(\mathscr{M}_2, \mathscr{N}_2) + \mathrm{Mas}(\mathscr{M}_7, \mathscr{N}_7) \\
&\quad = \mathrm{Mas}(L_{X,0,t}, L_{Y,0,0}) - \mathrm{Mas}(L_{X,0,t}, L_{Y,0,1}) \\
&\quad = \tau_\mu(JL_{X,0,1}, L_{Y,0,0}, L_{Y,0,1}) - \tau_\mu(JL_{X,0,0}, L_{Y,0,0}, L_{Y,0,1}).
\end{aligned}$$

We have $E^+_{0,\nu} = dE^-_{0,\nu} \oplus d^*E^-_{0,\nu}$, and we can compute

$$\begin{aligned}
&\tau_\mu(JL_{X,0,1}, L_{Y,0,0}, L_{Y,0,1}) \\
&\quad = \tau_\mu(E^+_{0,\nu} \oplus K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+, (W_Y \oplus JW_Y^\perp) \oplus dE^-_{0,\nu} \oplus \mathscr{L}_{Y,0}, E^+_{0,\nu} \oplus K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+) \\
&\quad = \tau_\mu(d^*E^-_{0,\nu} \oplus K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+, (W_Y \oplus JW_Y^\perp) \oplus \mathscr{L}_{Y,0}, d^*E^-_{0,\nu} \oplus K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+) \\
&\quad = \dim(JW_Y) + \dim(J\mathscr{L}_{Y,0} \cap (K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+)).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\tau_\mu(JL_{X,0,0}, L_{Y,0,0}, L_{Y,0,1}) \\
&\quad = \tau_\mu((JW_X \oplus W_X^\perp) \oplus d^*E^-_{0,\nu} \oplus J\mathscr{L}_{X,0}, (W_Y \oplus JW_Y^\perp) \oplus dE^-_{0,\nu} \oplus \mathscr{L}_{Y,0}, \\
&\qquad\qquad E^+_{0,\nu} \oplus K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+) \\
&\quad = \tau_\mu(d^*E^-_{0,\nu} \oplus J\mathscr{L}_{X,0}, (W_Y \oplus JW_Y^\perp) \oplus \mathscr{L}_{Y,0}, d^*E^-_{0,\nu} \oplus K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+) \\
&\quad = \dim(JW_Y) + \tau(J\mathscr{L}_{X,0}, \mathscr{L}_{Y,0}, K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+).
\end{aligned}$$

i	paths $\mathscr{M}_i(t)$	Endpoints of $\mathscr{M}_i$ and $\mathscr{N}_i$		paths $\mathscr{N}_i(t)$
		$\Lambda_{X,0}$	$\Lambda_{Y,0}$	
1	$\Lambda_{X,0}^{R_t}$			$\Lambda_{Y,0}^{R_t}$
		$\Lambda_{X,0}^{\infty}$	$\Lambda_{Y,0}^{\infty}$	
2	$P_{0,\nu}^{-} \oplus L_{X,0,t}$			$\Lambda_{Y,0}^{R_{1-t}}$
		$\mathscr{P}_{\tilde{\varrho}(0)}^{-}$	$\Lambda_{Y,0}$	
3	$\mathscr{P}_{\tilde{\varrho}(t)}^{-}$			$\Lambda_{Y,t}$
		$\mathscr{P}_{\tilde{\varrho}(1)}^{-}$	$\Lambda_{Y,1}$	
4	constant			$\Lambda_{Y,1}^{R_t}$
		$\mathscr{P}_{\tilde{\varrho}(1)}^{-}$	$\Lambda_{Y,1}^{\infty}$	
5	constant			$P_{1,\nu}^{+} \oplus L_{Y,1,t}$
		$\mathscr{P}_{\tilde{\varrho}(1)}^{-}$	$\mathscr{P}_{\tilde{\varrho}(1)}^{+}$	
6	$\mathscr{P}_{\tilde{\varrho}(1-t)}^{-}$			$\mathscr{P}_{\tilde{\varrho}(1-t)}^{+}$
		$\mathscr{P}_{\tilde{\varrho}(0)}^{-}$	$\mathscr{P}_{\tilde{\varrho}(0)}^{+}$	
7	$P_{0,\nu}^{-} \oplus L_{X,0,1-t}$			constant
		$\mathscr{P}_{\tilde{\varrho}(0)}^{-}$	$\mathscr{P}_{\tilde{\varrho}(0)}^{+}$	
8	$\Lambda_{X,0}^{R_{1-t}}$			constant
		$\Lambda_{X,0}$	$\mathscr{P}_{\tilde{\varrho}(0)}^{+}$	
9	$\Lambda_{X,t}$			$\mathscr{P}_{\tilde{\varrho}(t)}^{+}$
		$\Lambda_{X,1}$	$\mathscr{P}_{\tilde{\varrho}(1)}^{+}$	
10	$\Lambda_{X,1}^{R_t}$			$P_{1,\nu}^{+} \oplus L_{Y,1,1-t}$
		$\Lambda_{X,1}^{\infty}$	$\Lambda_{Y,1}^{\infty}$	
11	$\Lambda_{X,1}^{R_{1-t}}$			$\Lambda_{Y,1}^{R_{1-t}}$
		$\Lambda_{X,1}$	$\Lambda_{Y,1}$	

Table 1: The paths homotopic to $\Lambda_{X,t}$ and $\Lambda_{Y,t}$ broken up into pieces

Thus, together with [12, Proposition 6.11], this shows

$$\begin{aligned}\operatorname{Mas}(\mathscr{M}_2,\mathscr{N}_2)+\operatorname{Mas}(\mathscr{M}_7,\mathscr{N}_7) &= \dim(J\mathscr{L}_{Y,0}\cap(K^+_{\tilde\varrho(0)}\oplus\hat{\mathscr{L}}^+))\\ &\quad - \tau(J\mathscr{L}_{X,0},\mathscr{L}_{Y,0},K^+_{\tilde\varrho(0)}\oplus\hat{\mathscr{L}}^+)\\ &= \tau(J\mathscr{L}_{X,0},K^+_{\tilde\varrho(0)}\oplus\hat{\mathscr{L}}^+,\mathscr{L}_{Y,0}).\end{aligned}$$

Similarly we get

$$\operatorname{Mas}(\mathscr{M}_5,\mathscr{N}_5)+\operatorname{Mas}(\mathscr{M}_{10},\mathscr{N}_{10}) = -\tau(J\mathscr{L}_{X,1},K^+_{\tilde\varrho(1)}\oplus\hat{\mathscr{L}}^+,\mathscr{L}_{Y,1}).$$

This completes the proof. □

The ideal situation for applying Theorem 2.6 is when the manifold M splits into a solid torus $D^2\times S^1$ and its complement Y, and the path consists of connections that are flat on Y. When this is not the case, Theorem 2.6 can still provide some useful information. We start with a simple observation.

Lemma 2.7 *Let A_t and A'_t be loops of $SU(n)$ connections on 3–manifolds X and X', both with boundary the surface Σ , and let $\mathscr{P}_t$ a continuous family of boundary conditions that make D_{A_t} and $D_{A'_t}$ self-adjoint. Then*

$$\operatorname{SF}(A_t|_X,\mathscr{P}_t)=\operatorname{SF}(A'_t|_{X'},\mathscr{P}_t).$$

Proof Let Λ be a Lagrangian subspace, such that $(\Lambda,\mathscr{P}_t)$ is a Fredholm pair for all t. Then, by the contractibility of the space of connections we have

$$\operatorname{SF}(A_t|_X,\mathscr{P}_t)=\operatorname{Mas}(\Lambda_{X,A_t},\mathscr{P}_t)=\operatorname{Mas}(\Lambda,\mathscr{P}_t)=\operatorname{Mas}(\Lambda_{X',A'_t},\mathscr{P}_t)=\operatorname{SF}(A'_t|_{X'},\mathscr{P}_t).$$

□

Therefore, the spectral flow of the odd signature operator coupled to a loop of $SU(n)$ connections on a manifold with boundary only depends on its restriction to the boundary. Orient the solid torus S such that the orientations of S and X agree in a collar of $\partial S=\partial X$.

Definition 2.8 Given a loop $\tilde\varrho$ in $\widetilde{\Lambda^2}$ with projection ϱ in Λ^2, let A_t be a path of $SU(n)$ connections on the solid torus S restricting to $a_{\varrho(t)}$ on the boundary. We define $\operatorname{SF}(\tilde\varrho):=\operatorname{SF}(A_t|_S;\mathscr{P}^+_{\tilde\varrho(t)})$.

Since the spectral flow is a homotopy invariant and additive under concatenation of paths in $\widetilde{\Lambda^2}$, the computation for an arbitrary loop in $\widetilde{\Lambda^2}$ can be reduced to a loop $\tilde{\varrho} = (\alpha, \beta, \theta)$, where $(\alpha(t), \beta(t))$ is constant and lies in exactly one $\mathscr{Z}_{ij}$, and $\theta(t) = (\theta_{kl}(t))$ for $\theta_{kl}(t) = 1$ unless $k = i$ and $l = j$, in which case $\theta_{ij}(t) = e^{2\pi i t}$, $t \in [0, 1]$. After gauge transformation we may further assume that $(i, j) = (1, 2)$. Then, we can assume after homotopy that

$$(\alpha, \beta) \equiv ((\alpha_1, \alpha_2, 0, \dots, 0), (\beta_1, \beta_2, 0, \dots, 0)) \in \mathscr{Z}_{12}.$$

Consequently, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \frac{1}{2}\mathbb{Z}$. Let us identify $SU(2)$ with $SU(2) \times \{\mathrm{Id}\} \subset SU(n)$ and $su(2)$ with $su(2) \times \{0\} \subset su(n)$. Let ϱ be the projection of $\tilde{\varrho}$ in Λ^2, and let A_t be a path of $SU(2)$ connections on the solid torus S restricting to $a_{\varrho(t)}$ on the boundary. Then we compute

$$\begin{aligned} \mathrm{SF}(\tilde{\varrho}) &= \mathrm{SF}(A_t|_S; \mathscr{P}^+_{\tilde{\varrho}(t)}) \\ &= \mathrm{SF}(A_t|_S; P^{12+}_{\tilde{\varrho}(t)} \oplus (U_n\, dm \oplus U_n\, dm \wedge d\ell) \oplus K^{12+}_{\tilde{\varrho}(t)}). \end{aligned}$$

Since $U_n\, dm \oplus U_n\, dm \wedge d\ell$ is transverse to $U_n \oplus U_n\, d\ell$, we can apply [11, Theorem 5.3.3] to compute that $\mathrm{SF}(\tilde{\varrho}) = 4$.

We define the winding number for loops $\tilde{\varrho}$ in $\widetilde{\Lambda^2}$ as follows. First homotope $\tilde{\varrho}$ to a product $\tilde{\varrho}^1 * \cdots * \tilde{\varrho}^m$ of loops such that each $\tilde{\varrho}^k = \tilde{\tau}^k * (\alpha^k, \beta^k, \theta^k) * (\tilde{\tau}^k)^{-1}$ with $(\alpha^k(t), \beta^k(t))$ constant. Then we define

$$\mathrm{wind}(\tilde{\varrho}) := \sum_{k=1}^{m} \sum_{\substack{(i,j) \\ (\alpha^k, \beta^k) \in \mathscr{Z}_{ij}}} \mathrm{wind}\left(\theta^k_{ij}(t)\right).$$

Let us summarize.

Proposition 2.9 *Let $\tilde{\varrho}(t)$ be a loop in $\widetilde{\Lambda^2}$. Then*

$$\mathrm{SF}(\tilde{\varrho}) = 4\ \mathrm{wind}(\tilde{\varrho}).$$

Now we can state the main splitting formula.

Theorem 2.10 *Consider two flat connections B_0 and B_1 on $M = X \cup_T Y$. Let A_t and A'_t be paths of $SU(n)$ connections on X and Y, respectively, with $B_\varepsilon|_X = A_\varepsilon$ and $B_\varepsilon|_Y = A'_\varepsilon$, $\varepsilon = 0, 1$, satisfying the properties in Theorem 2.6 with $\tilde{\varrho}$ and $\tilde{\varrho}'$ the corresponding paths in $\widetilde{\Lambda^2}$. Then*

$$\begin{aligned} \mathrm{SF}(B_0, B_1) = &\,\mathrm{SF}(A_t; \mathscr{P}^+_{\tilde{\varrho}(t)}) + \mathrm{SF}(A'_t; \mathscr{P}^-_{\tilde{\varrho}'(t)}) + \mathrm{SF}(\tilde{\varrho}(1-t) * \tilde{\varrho}'(t)) \\ &+ \tau_\mu(J\mathscr{L}_{X,0}, K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,0}) - \tau_\mu(J\mathscr{L}_{X,1}, K^+_{\tilde{\varrho}(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,1}). \end{aligned}$$

Proof Extend A'_t arbitrarily to a path B_t from B_0 to B_1. Then

$$\begin{aligned}\mathrm{SF}(B_t) &= \mathrm{SF}(B_t|_X; \mathscr{P}^+_{\tilde{\varrho}'(t)}) + \mathrm{SF}(B_t|_Y; \mathscr{P}^-_{\tilde{\varrho}'(t)}) + \mathrm{SF}(A_t; \mathscr{P}^+_{\tilde{\varrho}(t)}) - \mathrm{SF}(A_t; \mathscr{P}^+_{\tilde{\varrho}(t)}),\\ &\quad + \tau_\mu(J\mathscr{L}_{X,0}, K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,0}) - \tau_\mu(J\mathscr{L}_{X,1}, K^+_{\tilde{\varrho}(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,1})\\ &= \mathrm{SF}(A_t; \mathscr{P}^+_{\tilde{\varrho}(t)}) + \mathrm{SF}(A'_t; \mathscr{P}^-_{\tilde{\varrho}'(t)}) + \mathrm{SF}(A_{1-t} * B_t|_X; \mathscr{P}^+_{\tilde{\varrho}(1-t)*\tilde{\varrho}'(t)})\\ &\quad + \tau_\mu(J\mathscr{L}_{X,0}, K^+_{\tilde{\varrho}(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,0}) - \tau_\mu(J\mathscr{L}_{X,1}, K^+_{\tilde{\varrho}(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{Y,1}).\end{aligned}$$

With Lemma 2.7 the desired formula follows. □

$\mathrm{SF}(\tilde{\varrho})$ can be defined for paths other than loops. Some examples have been computed in the case $n = 2$ by Himpel [11, Theorem 5.3.3].

3 The *SU(3)* representation variety of a spliced sum

Suppose K_1 and K_2 are knots in S^3 with complements $X_1 = S^3 \backslash \nu K_1$ and $X_2 = S^3 \backslash \nu K_2$, and let $M = X_1 \cup_T X_2$ be the spliced sum. In this section, we establish some basic results about the representation variety $R(M, SU(3))$.

Given a representation $\alpha\colon \pi_1(M) \to SU(3)$, we set $\alpha_1 = \alpha|_{\pi_1(X_1)}, \alpha_2 = \alpha|_{\pi_1(X_2)}$, and $\alpha_0 = \alpha|_{\pi_1(T)}$, and we will sometimes write $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$.

Lemma 3.1 *If $\alpha\colon \pi_1(M) \to SU(3)$ is a representation with α_1 or α_2 abelian, then α is trivial.*

Remark This lemma is true in general for representations $\alpha\colon \pi_1(M) \to SU(n)$, where M is the spliced sum along knots in S^3, but not for spliced sums along knots in homology spheres.

Proof Suppose α_1 is abelian. Because λ_1 lies in the commutator subgroup, it follows that $\alpha(\lambda_1) = I$. Splicing identifies μ_2 with λ_1, and it follows that $\alpha(\mu_2) = I$. Because μ_2 normally generates $\pi_1(X_2)$, we conclude that α_2 is trivial. In particular $\alpha(\lambda_2) = I$, and splicing again shows $\alpha(\mu_1) = I$ and the same argument shows α_1 is also trivial. □

Lemma 3.2 *If $\alpha\colon \pi_1(M) \to SU(3)$ is a representation with $\alpha(\mu_1)$ or $\alpha(\mu_2)$ central, then α is trivial.*

Proof Suppose $\alpha(\mu_1)$ is central. Since μ_1 normally generates $\pi_1(X_1)$, it follows that α_1 is abelian, and we apply Lemma 3.1 to make the conclusion. □

Because $\pi_1(T) = \mathbb{Z}^2$ is abelian, we can conjugate α so that both α_0 is diagonal. Thus, the stablizer subgroup $\mathrm{Stab}(\alpha_0)$ must contain the maximal torus $T_{SU(3)} \cong T^2$. The next two results show that, for the purposes of computing the $SU(3)$ Casson invariant, we can restrict our attention to representations with $\mathrm{Stab}(\alpha_0) = T_{SU(3)}$.

Proposition 3.3 *If $\alpha\colon \pi_1(M) \to SU(3)$ is a nontrivial representation with $\mathrm{Stab}(\alpha_0) \neq T_{SU(3)}$, then α_1 and α_2 are both irreducible.*

Proof Since $\pi_1(T) = \mathbb{Z}^2$ is abelian, we can conjugate α so that $\alpha(\mu_1)$ and $\alpha(\lambda_1)$ are both diagonal. Now if either of these elements has three distinct eigenvalues, then $\mathrm{Stab}(\alpha_0) = T_{SU(3)}$. Thus our hypotheses imply that $\alpha(\mu_1)$ and $\alpha(\mu_2)$ both have a double eigenvalue. If their 2–dimensional eigenspaces do not coincide, then we can find integers k, l such that the diagonal matrix $\alpha(\mu_1^k \lambda_1^l)$ has three distinct eigenvalues, and it would then follow that $\mathrm{Stab}(\alpha_0) = T_{SU(3)}$. Thus, we can assume that, up to conjugation,

$$\alpha(\mu_1) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \bar{a}^2 \end{pmatrix} \quad \text{and} \quad \alpha(\lambda_1) = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \bar{b}^2 \end{pmatrix}$$

for some $a, b \in U(1)$ not equal to a third root of unity.

Now suppose to the contrary that α_1 is reducible. Then, up to conjugation, α_1 has image in $S(U(2) \times U(1))$. Since λ_1 lies in the commutator subgroup of $\pi_1(X_1)$, its image under α must lie in the commutator group of $S(U(2) \times U(1))$, which is $SU(2) \times \{1\}$. This shows that one of the eigenvalues of $\alpha(\lambda_1)$ must equal 1. If $b = 1$, then $\alpha(\mu_2) = \alpha(\lambda_1) = I$ and Lemma 3.2 implies α is trivial, a contradiction. Otherwise, $b^2 = 1$ and $b = -1$ and we see then that $\alpha(\mu_1)$ lies in the center of $\alpha_1(\pi_1(X_1))$. Because μ_1 normally generates this group, this shows that α_1 is abelian and Lemma 3.1 gives the desired contradiction. □

For further results, we need to make the additional assumptions that the representation varieties $R(X_1, SU(3))$ and $R(X_2, SU(3))$ are in general position in the "$SU(3)$ pillowcase" $R(T, SU(3))$. Specifically, we assume that the images of $R(X_1, SU(3))$ and $R(X_2, SU(3))$ intersect transversely in $R(T, SU(3))$, and that the restriction maps

$$R(X_1, SU(3)) \to R(T, SU(3)) \quad \text{and} \quad R(X_2, SU(3)) \to R(T, SU(3))$$

are both local immersions in a neighborhood of each intersection point. These assumptions will not hold in general for spliced sums along knots in S^3, but one can check that they do hold for spliced sums along $(2, q)$ torus knots.

In the following result, we use $[\alpha]$ to denote the conjugacy class of a representation $\alpha\colon \pi_1(M) \to SU(3)$.

Proposition 3.4 *Suppose the above transversality assumption holds for all representations $\alpha\colon \pi_1(M) \to SU(3)$ and suppose α is nontrivial with $\mathrm{Stab}(\alpha_0) \neq T_{SU(3)}$. Set*

$$C = \{[\beta] \in R(M, SU(3)) \mid \ \beta_i \text{ is conjugate to } \alpha_i \text{ for } i = 1, 2\}.$$

Then $C \subset R^*(M, SU(3))$ and is diffeomorphic to $S(U(2) \times U(1))/Z_{SU(3)}$, where $Z_{SU(3)} \cong \mathbb{Z}_3$ is the center of $SU(3)$. In particular, we have $\chi(C) = 0$.

Proof Proposition 3.3 implies that C consists entirely of irreducible representations, and under the transversality assumption, this component can be described as the double coset $\Gamma_1 \backslash \Gamma_0 / \Gamma_2$, where $\Gamma_i = \mathrm{Stab}(\alpha_i)$. Proposition 3.3 shows that $\Gamma_1 = \Gamma_2 = Z_{SU(3)}$, and its proof shows that $\Gamma_0 = S(U(2) \times U(1))$. Since $S(U(2) \times U(1))$ is diffeomorphic to $U(2)$, it has zero Euler characteristic. □

If $\alpha\colon \pi_1(M) \to SU(3)$ is a nontrivial representation with $\mathrm{Stab}(\alpha_0) = T_{SU(3)}$, then we have exactly three possibilities:

(a) Both α_1 and α_2 are irreducible,

(b) One of α_1, α_2 is irreducible, the other is reducible and nonabelian, or

(c) Both α_1 and α_2 are reducible and nonabelian.

The next result shows that, for the purposes of computing the $SU(3)$ Casson invariant of spliced sums, the only contributions come from case (c).

Proposition 3.5 *Let the above assumption hold for all representations $\alpha\colon \pi_1(M) \to SU(3)$, and suppose α is a nontrivial representation with $\mathrm{Stab}(\alpha_0) = T_{SU(3)}$ and one of α_1 or α_2 irreducible. (So we are in case (a) or case (b).) Set*

$$C = \{[\beta] \in R(M, SU(3)) \mid \ \beta_i \text{ is conjugate to } \alpha_i \text{ for } i = 1, 2\}.$$

Then $C \subset R^*(M, SU(3))$ with $C \cong T_{SU(3)}/Z_{SU(3)}$ in Case (a) and $C \cong T_{SU(3)}/U(1)$ in case (b). In either case, we see that $\chi(C) = 0$.

Proof Using the double coset description of the component, we see that $C = \Gamma_1 \backslash \Gamma_0 / \Gamma_2$ where $\Gamma_0 = T_{SU(3)}$. In case (a), we get that $\Gamma_1 = \Gamma_2 = Z_{SU(3)}$ and the first result follows. In case (b), assuming (wlog) that α_1 is irreducible and α_2 is reducible, we find that $\Gamma_1 = Z_{SU(3)}$ and

$$\Gamma_2 = \left\{ \begin{pmatrix} e^{\theta i} & 0 & 0 \\ 0 & e^{\theta i} & 0 \\ 0 & 0 & e^{-2\theta i} \end{pmatrix} \middle| \theta \in [0, 2\pi] \right\} \cong U(1),$$

and the second result follows. □

The only remaining case is Case (c), where both α_1 and α_2 are reducible and non-abelian. There are two possibilities here:

(c-1) Both α_1 and α_2 can be simultaneously conjugated to lie in $S(U(2) \times U(1)$. In this case, $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ is reducible and lies on a component $C \cong S^1$ consisting entirely of reducible representations.

(c-2) After conjugating, α_1 lies in $S(U(2) \times U(1))$ and α_2 lies in $S(U(1) \times U(2))$. In this case $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ is irreducible and its conjugacy class $[\alpha]$ is an isolated point in $R^*(M, SU(3))$.

The next result summarizes our discussion and gives a classification of the possible components of $R(M, SU(3))$ for spliced sums satisfying the transversality assumption.

Theorem 3.6 *Suppose M is a spliced sum along knots in S^3 and satisfies the transversality assumption. Then the representation variety $R(M, SU(3)) = \bigcup_{j \in J} C_j$ is a disjoint union of components C_j that are either entirely contained in $R^*(M, SU(3))$ or disjoint from $R^*(M, SU(3))$. If $C_j \subseteq R^*(M, SU(3))$, then C_j is diffeomorphic to one of*

$$S(U(2) \times U(1))/Z_{SU(3)}, \quad T_{SU(3)}/Z_{SU(3)}, \quad T_{SU(3)}/U(1), \quad \{*\},$$

depending on the level of reducibility of $\alpha_0, \alpha_1, \alpha_2$. Otherwise, if $C_j \cap R^*(M, SU(3)) = \varnothing$, then C_j is diffeomorphic to S^1 or $\{*\}$, the latter occurring only when $C_j = \{[\Theta]\}$, the trivial representation.

Remark Notice that the positive dimensional components C_j all satisfy $\chi(C_j) = 0$. Using the homeomorphism between the moduli space $\mathscr{M}$ of flat $SU(3)$ connections on M and the representation variety $R(M, SU(3))$ provided by the holonomy map, the transversality assumption ensures that each of the corresponding components in $\mathcal{M}$ is a nondegenerate critical submanifold for the Chern–Simons function. In particular, by [2, Theorem 7], we see that these components do not contribute to the $SU(3)$ Casson invariant. In order to compute the $SU(3)$ Casson invariant, in Section 5 we will concentrate on the 0–dimensional or isolated components.

Notice further that for components of type (c-2), which are the isolated points of $R^*(M, SU(3))$, it is possible to have α_1 conjugate to α_1' as $S(U(2) \times U(1))$ representations of $\pi_1(X_1)$, and α_2 conjugate to α_2' as $S(U(1) \times U(2))$ representations of $\pi_1(X_2)$, but $\alpha_1 \cup_{\alpha_0} \alpha_2$ not conjugate to $\alpha_1' \cup_{\alpha_0'} \alpha_2'$ as $SU(3)$ representations of $\pi_1(M)$ for the spliced sum $M = X_1 \cup_T X_2$. This is a consequence of the existence of discrete gluing parameters in this context, and we will return to this issue in Theorem 5.1, where we enumerate the isolated components of $R^*(M, SU(3))$ for certain spliced sums.

4 *SU(3)* representation varieties of knot complements

In the previous section, we examined the $SU(3)$ representation varieties of spliced sums and discovered that the only contributions to the $SU(3)$ Casson invariant come from representations $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ with α_1 and α_2 reducible, nonabelian representations of the knot complements. In this section, we study the representation varieties $R(X, SU(3))$ for knot complements. In general, $R(X, SU(3))$ is a union of three different strata:

1. $R^*(X, SU(3))$ the stratum of irreducible representations,
2. $R^{red}(X, SU(3))$ the stratum of reducible nonabelian representations, and
3. $R^{ab}(X, SU(3))$ the stratum of abelian representations.

Because our computations of $\lambda_{SU(3)}(M)$ for spliced sums involve only those representations that restrict to reducible, nonabelian representations on X_1 and X_2, we concentrate on the stratum $R^{red}(X, SU(3))$. We shall use the results of Klassen [13] to give a useful description in case X is the complement of a $(2, q)$ torus knot. The curious reader is referred to [4, Section 3] for descriptions of the other strata. The results presented here are complementary to those in [4].

Let K be the $(2, q)$ torus knot and $X = S^3 \setminus \nu K$ its complement. The knot group $\pi_1(X)$ has presentation

$$\pi_1(X) \cong \langle x, y \mid x^2 = y^q \rangle, \tag{4–1}$$

with meridian $\mu = xy^{\frac{1-q}{2}}$ and longitude $\lambda = x^2 \mu^{-2q}$.

Every reducible representation $\alpha\colon\ \pi_1(X) \to SU(3)$ can be conjugated to lie in $S(U(2) \times U(1))$. Furthermore, every $S(U(2) \times U(1))$ representation of $\pi_1(X)$ is obtained by twisting an $SU(2)$ representation. In [13], Klassen proves that $R^*(X, SU(2))$ is a union of $q - 1$ open arcs, and using this, we shall show that $R^{red}(X, SU(3))$ is a union of $q - 1$ open Möbius bands.

In the next result, we identify $SU(2)$ with the unit quaternions by the map

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + bj \quad \text{for } a, b \in \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1.$$

To each $t \in [0, \frac{1}{2}]$ we associate the abelian representation $\beta_t : \pi_1(X) \to SU(2)$ with $\beta_t(\mu) = e^{2\pi i t}$. In this way, we parameterize $R^{ab}(X, SU(2))$ by the closed interval $[0, \frac{1}{2}]$.

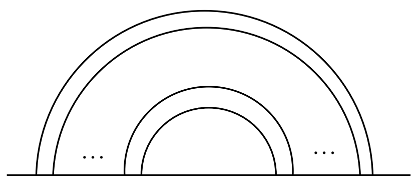

Figure 2: The $SU(2)$ representation variety of a $(2, q)$ torus knot

Proposition 4.1 (Klassen) *The representation variety $R^*(X, SU(2))$ consists of $(q-1)/2$ open arcs given as follows. For $k \in \{1, 3, \ldots, q-2\}$ and $s \in [0, 1]$, define $\beta_{k,s}$ by setting*

$$\begin{aligned} \beta_{k,s}(x) =& i\cos(\pi s) + j\sin(\pi s), \\ \beta_{k,s}(y) =& \cos(\pi k/q) + i\sin(\pi k/q) = e^{k\pi i/q}. \end{aligned}$$

Then the resulting paths of $SU(2)$ representations $\beta_{k,s}$ are irreducible and have $H^1(X; su(2)_{\beta_{k,s}}) = \mathbb{R}$ and $H^1(Z; \mathbb{C}^2_{\beta_{k,s}}) = 0$ for $s \in (0, 1)$.

When $s = 0, 1$ the representations $\beta_{k,0}$ and $\beta_{k,1}$, are abelian with

$$\beta_{k,0}(\mu) = (-1)^{\frac{k-1}{2}} e^{\frac{k\pi i}{2q}} \quad \text{and} \quad \beta_{k,1}(\mu) = (-1)^{\frac{k+1}{2}} e^{\frac{k\pi i}{2q}}.$$

Using $[0, \frac{1}{2}]$ to parameterize the abelian representations, we see that the arc $\beta_{k,s}$ is attached at the bifurcation points $\left\{\frac{k}{4q}, \frac{2q-k}{4q}\right\}$ (see Figure 2).

Observe that the image of the meridian is given by

$$\beta_{k,s}(\mu) = (i\cos(\pi s) + j\sin(\pi s))e^{k\pi i\left(\frac{1-q}{2q}\right)},$$

and a quick calculation shows that $\beta_{k,s}(\mu)$ is conjugate to the diagonal matrix

$$\begin{pmatrix} e^{2u\pi i} & 0 \\ 0 & e^{-2u\pi i} \end{pmatrix},$$

where $u \in [0, \frac{1}{2}]$ satisfies

$$\cos(2\pi u) = \cos(\pi s)\sin\left(\frac{k(q-1)\pi}{2q}\right).$$

Since $s \in [0,1]$ and

$$\sin\left(\tfrac{k(q-1)\pi}{2q}\right) = \sin\left(k\left(\tfrac{\pi}{2} - \tfrac{\pi}{2q}\right)\right) = (-1)^{(k-1)/2}\cos\left(\tfrac{k\pi}{2q}\right),$$

we see that

$$u \in \left(\tfrac{k}{4q}, \tfrac{2q-k}{4q}\right). \tag{4–2}$$

Since $\lambda = x^2\mu^{-2q}$, then $\beta_{k,s}(\lambda)$ is conjugate to

$$\begin{pmatrix} -e^{-2q(2u\pi i)} & 0 \\ 0 & -e^{2q(2u\pi i)} \end{pmatrix}.$$

We are interested in the restriction of $\beta_{k,s}$ to the boundary torus. Recall that $R(T, SU(2))$ is modelled by the pillowcase, which is the quotient of the 2-torus T^2 by the involution sending (x,y) to $(1-x, 1-y)$, where we think of T^2 as $[0,1]\times[0,1]$ with opposite sides identified. Under this identification, the point $(u,v) \in [0,\frac{1}{2}]\times[0,1]$ in the pillowcase corresponds to the diagonal representation $\beta\colon\ \pi_1(T) \to SU(2)$ with

$$\beta(\mu) = \begin{pmatrix} e^{2u\pi i} & 0 \\ 0 & e^{-2u\pi i} \end{pmatrix} \text{ and } \beta(\lambda) = \begin{pmatrix} e^{2v\pi i} & 0 \\ 0 & e^{-2v\pi i} \end{pmatrix}.$$

For $s \in [0,1]$, the restriction of $\beta_{k,s}$ to the boundary torus gives a line of slope $-2q$ in the pillowcase connecting $\left(\frac{k}{4q}, 0\right)$ to $\left(\frac{2q-k}{4q}, 0\right)$ and wrapping around vertically $q-k$ times.

Using the *twisting* operation [4, § 3.2], we give an explicit description of $R^{red}(X, SU(3))$ as a union of $(q-1)/2$ Möbius bands, which are 2–dimensional families obtained by twisting the arcs $\beta_{k,s}$ by characters $\chi : \pi_1(X) \to U(1)$.

First, in terms of matrices, if $A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in SU(2)$ and $e^{i\theta} \in U(1)$, we define the twist of A by $e^{i\theta}$ to be the $S(U(2)\times U(1))$ matrix

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} \begin{pmatrix} a & b & 0 \\ -\overline{b} & \overline{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta}a & e^{i\theta}b & 0 \\ -e^{i\theta}\overline{b} & e^{i\theta}\overline{a} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}.$$

Given an irreducible representation $\beta\colon\ \pi_1(X) \to SU(2)$ and an abelian representation $\chi : \pi_1(X) \to U(1)$, we define the reducible $SU(3)$ representation obtained by twisting β by χ, denoted $\chi \odot \beta$, to be the $S(U(2)\times U(1))$ representation taking an element $\gamma \in \pi_1(X)$ to the twist of $\beta(\gamma)$ by $\chi(\gamma)$.

Since abelian representations factor through the homology group $H_1(X, \mathbb{Z})$, which is generated by the meridian μ, we see that a representation $\chi\colon\ \pi_1(X) \to U(1)$ is determined by $\chi(\mu)$.

Definition 4.2 For $e^{i\theta} \in U(1)$, let χ_θ be the $U(1)$ representation with $\chi_\theta(\mu) = e^{i\theta}$. For $k \in \{1, 3, \dots, q-2\}$ and $s \in (0,1)$, let $\beta_{k,s}$ be the $SU(2)$ representation described in Proposition 4.1 and define $\alpha_{k,s,\theta} = \chi_\theta \odot \beta_{k,s}$ to be the reducible $SU(3)$ representation obtained by twisting $\beta_{k,s}$ by χ_θ.

Notice that if $\theta = \pi$, the twist of an $SU(2)$ representation β by χ_π takes values in the $SU(2) \times \{1\}$ matrices, and a quick calculation shows that

$$\chi_\pi \odot \beta_{k,s} \text{ is conjugate to } \beta_{k,1-s}. \tag{4–3}$$

Thus, for $k \in \{1, 3, \dots, q-2\}$, the 2–dimensional family $\alpha_{k,s,\theta}$ is parameterized by $(s,\theta) \in (0,1) \times [0,\pi]$ with identification $(s,0) \sim (1-s,\pi)$. This gives an open Möbius band. The next result summarizes our discussion.

Proposition 4.3 *If X is the complement of the $(2,q)$ torus knot, then $R^{red}(X, SU(3))$ is a union of $\frac{q-1}{2}$ open Möbius bands. The closure of each stratum intersects the abelian stratum $R^{ab}(X, SU(3))$ in an immersed circle with isolated double points.*

5 Isolated components of $R^*(M, SU(3))$

In this section, we enumerate the isolated components in $R^*(M, SU(3))$ for M the spliced sum along torus knots of type $(2,q_1)$ and a $(2,q_2)$. Let K_1 and K_2 be $(2,q_1)$ and $(2,q_2)$ torus knots with complements X_1 and X_2, and write $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ according to the decomposition $M = X_1 \cup_T X_2$. Assume that $[\alpha]$ is isolated. By Section 3, we can assume that α is irreducible and both α_1 and α_2 are reducible. These are the type (c-2) components from Section 3, and they are the only components that contribute nontrivially to the $SU(3)$ Casson invariant. Note further that such a representation can be conjugated so that α_1 reduces to $S(U(2) \times U(1))$, α_2 reduces to $S(U(1) \times U(2))$, and α_0 is diagonal.

We can describe α_1 as the twist of an $SU(2)$ representation β_1 by a character χ_{θ_1}, and we get a similar statement for α_2 using the following refinement of twisting. For this purpose, we set $\odot_1 = \odot$ and define $\odot_2$ to be the twisting induced by the map which, for $e^{i\theta} \in U(1)$ and $A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$, gives the $S(U(1) \times U(2))$ matrix

$$\begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{-i\theta}a & e^{-i\theta}b \\ 0 & -e^{-i\theta}\bar{b} & e^{-i\theta}\bar{a} \end{pmatrix}.$$

On the level of representations, if $\beta_2\colon\ \pi_1(X_2) \to SU(2)$ and $\chi_{\theta_2}\colon\ \pi_1(X_2) \to U(1)$, then set $\chi_2 \odot_2 \beta_{\theta_2}$ to be the $S(U(1) \times U(2))$ representation obtained by twisting β_2 by χ_{θ_2} in this way. Assume now $\alpha_1 = \chi_{\theta_1} \odot_1 \beta_1$ and $\alpha_2 = \chi_{\theta_2} \odot_2 \beta_2$ for $SU(2)$ representations β_1, β_2 and characters $\chi_{\theta_1}, \chi_{\theta_2}$

Remark Note that, by Lemma 3.1, we can assume that β_1 and β_2 are both irreducible since $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ is irreducible.

The pair $\alpha_1\colon\ \pi_1(X_1) \to S(U(2) \times U(1))$ $\alpha_2\colon\ \pi_1(X_2) \to S(U(1) \times U(2))$ will extend to a representation $\alpha\colon\ \pi_1(M) \to SU(3)$ if and only if their restrictions to $\pi_1(T)$ agree, namely if and only if $\alpha_1(\mu_1) = \alpha_2(\lambda_2)$ and $\alpha_2(\mu_2) = \alpha_1(\lambda_1)$.

Theorem 5.1 *Suppose M is the spliced sum along torus knots K_1 and K_2 of type $(2, q_1)$ and $(2, q_2)$. Then the number of isolated conjugacy classes in $R^*(M, SU(3))$ is given by*

$$16\lambda'_{SU(2)}(K_1)\lambda'_{SU(2)}(K_2) = \frac{(q_1^2 - 1)(q_2^2 - 1)}{4},$$

where $\lambda'_{SU(2)}(K) = \Delta''_K(1)$ is the $SU(2)$ Casson knot invariant.

Proof Using equation (4–1) and the splice relations, we find that $\pi_1(M)$ has presentation

$$\pi_1(M) = \langle x_1, y_1, x_2, y_2 \mid x_1^2 = y_1^{q_1}, x_2^2 = y_2^{q_2}, \mu_1 = \lambda_2, \lambda_1 = \mu_2 \rangle,$$

where $\mu_1 = x_1 y_1^{\frac{q_1 - 1}{2}}, \lambda_1 = x_1^2 \mu_1^{-2q_1}$ and $\mu_2 = x_2 y_2^{\frac{q_2 - 1}{2}}, \lambda_2 = x_2^2 \mu_2^{-2q_2}$. Assume $\alpha = \alpha_1 \cup_{\alpha_0} \alpha_2$ is an irreducible representation of $\pi_1(M)$ with α_1 and α_2 both reducible, and conjugate so that α_1 is in $S(U(2) \times U(1))$ and α_2 is in $S(U(1) \times U(2))$.

Because the longitude λ_1 lies in the commutator subgroup of $\pi_1(X_1)$, reducibility of α_1 implies that $\alpha_1(\lambda_1)$ must have a 1 in the lower right-hand corner. Similarly, because λ_2 lies in the commutator subgroup of $\pi_1(X_2)$, reducibility of α_2 implies that $\alpha_2(\lambda_2)$ must have a 1 in the upper left-hand corner. Notice that twisting does not alter the image of the longitude since $\chi_{\theta_i}(\lambda_i) = 1$ for any $\theta_i \in [0, \pi]$. Thus, if $\alpha_1 = \chi_{\theta_1} \odot_1 \beta_1$ and $\alpha_2 = \chi_{\theta_2} \odot_2 \beta_2$, then the only way to have a 1 in the upper right-hand corner of $\alpha_1(\mu_1)$ and also in the lower right-hand corner of $\alpha_2(\mu_2)$ is if

$$\beta_1(\mu_1) = \begin{pmatrix} e^{-\theta_1 i} & 0 \\ 0 & e^{\theta_1 i} \end{pmatrix} \quad \text{and} \quad \beta_2(\mu_2) = \begin{pmatrix} e^{-\theta_2 i} & 0 \\ 0 & e^{\theta_2 i} \end{pmatrix}.$$

In that case,

$$\beta_1(\lambda_1) = \begin{pmatrix} -e^{2q_1\theta_1 i} & 0 \\ 0 & -e^{-2q_1\theta_1 i} \end{pmatrix} \quad \text{and} \quad \beta_2(\lambda_2) = \begin{pmatrix} -e^{2q_2\theta_2 i} & 0 \\ 0 & -e^{-2q_2\theta_2 i} \end{pmatrix}.$$

If $\alpha_1 = \chi_{\theta_1} \odot_1 \beta_1$ and $\alpha_2 = \chi_{\theta_2} \odot_2 \beta_2$, an easy compuation shows

$$\alpha_1(\mu_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\theta_1 i} & 0 \\ 0 & 0 & e^{-2\theta_1 i} \end{pmatrix}, \quad \alpha_1(\lambda_1) = \begin{pmatrix} -e^{2q_1\theta_1 i} & 0 & 0 \\ 0 & -e^{-2q_1\theta_1 i} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5–1}$$

$$\alpha_2(\mu_2) = \begin{pmatrix} e^{2\theta_2 i} & 0 & 0 \\ 0 & e^{-2\theta_2 i} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha_2(\lambda_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e^{2q_2\theta_2 i} & 0 \\ 0 & 0 & -e^{-2q_2\theta_2 i} \end{pmatrix}. \tag{5–2}$$

The results of the previous section imply that β_1 and β_2 are conjugate to representations β_{k_1,s_1} and β_{k_2,s_2} of Proposition 4.1 for some $k_1 = 1, 3, \ldots, q_1 - 2$ and $k_2 = 1, 3, \ldots, q_2 - 2$ and $s_1, s_2 \in (0,1)$. As noted in Section 4, $\beta_{k_1,s_1}(\mu_1)$ and $\beta_{k_2,s_2}(\mu_2)$ are conjugate to

$$\begin{pmatrix} e^{-2u_1\pi i} & 0 \\ 0 & e^{2u_1\pi i} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{-2u_2\pi i} & 0 \\ 0 & e^{2u_2\pi i} \end{pmatrix},$$

respectively, where u_1, u_2 satisfy

$$\cos(2\pi u_1) = \cos(\pi s_1)\sin\left(\tfrac{k_1(q_1-1)\pi}{2q_1}\right) \text{ and } \cos(u_2) = \cos(\pi s_2)\sin\left(\tfrac{k_2(q_2-1)\pi}{2q_2}\right)$$

and $u_1 \in \left(\frac{k_1}{4q_1}, \frac{2q_1-k_1}{4q_1}\right)$ and $u_2 \in \left(\frac{k_2}{4q_2}, \frac{2q_2-k_2}{4q_2}\right)$.

Fix k_1 and k_2 as above and set $\theta_1 = 2\pi u_1$ and $\theta_2 = 2\pi u_2$. Consider the two paths $\alpha_{1,s_1} = \chi_{\theta_1} \odot_1 \beta_{k_1,s_1}$ and $\alpha_{2,s_2} = \chi_{\theta_2} \odot_2 \beta_{k_2,s_2}$ of reducible $SU(3)$ representations defined for $s_1, s_2 \in (0,1)$. (We conjugate β_{k_1,s_1} and β_{k_2,s_2} so that $\beta_{k_1,s_1}(\mu_1)$ and $\beta_{k_2,s_2}(\mu_2)$ are both diagonal in $SU(2)$.) Notice that the upper left-hand entry of $\alpha_{1,s_1}(\mu_1)$ is always equal to 1, as is the lower right-hand entry of $\alpha_{2,s_2}(\mu_2)$.

Consider the two arcs in T^2 defined in terms of α_{1,s_1} and α_{2,s_2} as follows. The first arc has its first coordinate given by the $(2,2)$ entry of $\alpha_{1,s_1}(\mu_1)$ and its second coordinate given by the $(1,1)$ entry of $\alpha_{1,s_1}(\lambda_1)$. The second arc has its first coordinate given by the $(2,2)$ entry of $\alpha_{2,s_2}(\lambda_2)$ and its second coordinate given by the $(1,1)$ entry of $\alpha_{2,s_2}(\mu_2)$. By (5–1) and (5–2), we see that the first arc is given by $(e^{2\theta_1 i}, -e^{2q_1\theta_1 i})$ for $\theta_1 \in \left(\frac{k_1\pi}{2q_1}, \frac{(2q_1-k_1)\pi}{2q_1}\right)$, whereas the second arc is given by $(-e^{2q_2\theta_2 i}, e^{2\theta_2 i})$ for $\theta_2 \in \left(\frac{k_2\pi}{2q_2}, \frac{(2q_2-k_2)\pi}{2q_2}\right)$.

Using γ_1 and γ_2 to denote the resulting curves in T^2, notice that γ_1 has slope q_1 and wraps around the 2-torus vertically $q_1 - k_1$ times, whereas γ_2 has slope $\frac{1}{q_2}$ and wraps around the 2-torus horizontally $q_2 - k_2$ times. From this, one sees that γ_1 and γ_2 intersect in $(q_1-k_1)(q_2-k_2)$ points. (One can perform the computation in homology by adding a horizontal segment to γ_1 that misses γ_2 and a vertical segment to γ_2 that misses γ_1, creating 1–cycles in the pillowcase minus the corners.)

Of course, the intersection points of γ_1 and γ_2 exactly coincide with choices of α_{1,s_1} and α_{2,s_2} that extend to an irreducible $SU(3)$ representation of $\pi_1(M)$, and each of these is an isolated point in $R^*(M, SU(3))$.

Summing over $k_1 \in \{1, 3, \ldots, q_1 - 2\}$ and $k_2 \in \{1, 3, \ldots, q_2 - 2\}$ and setting $j_1 = \frac{k_1-1}{2}$ and $j_2 = \frac{k_2-1}{2}$, we compute that

$$\sum_{j_1=1}^{\frac{q_1-1}{2}} \sum_{j_2=1}^{\frac{q_2-1}{2}} (q_1 - 2j_1 + 1)(q_2 - 2j_2 + 1) = \left(\sum_{j_1=1}^{\frac{q_1-1}{2}} q_1 - 2j_1 + 1 \right) \left(\sum_{j_2=1}^{\frac{q_2-1}{2}} q_2 - 2j_2 + 1 \right)$$
$$= \frac{(q_1^2 - 1)(q_2^2 - 1)}{16}.$$

We now take into account the fact that the conjugacy class of $\alpha_1 \cup_{\alpha_0} \alpha_2$ on $M = X_1 \cup_T X_2$ is not determined by the conjugacy classes of α_1 on X_1 and α_2 on X_2 (see the Remark following Theorem 3.6). Suppose as above $\alpha_0 \colon \pi_1(T) \to SU(3)$ is abelian with $\mathrm{Stab}(\alpha_0) = T_{SU(3)}$, the maximal torus, and consider the effect of conjugating by an element in $SU(3)$ that normalizes $T_{SU(3)}$. (Recall $N_{T_{SU(3)}}/Z_{T_{SU(3)}} \cong S_3$, the symmetric group on three letters.) On X_1, we further require that the conjugating element preserve $S(U(2) \times U(1))$, and on X_2 that it preserve $S(U(1) \times U(2))$. Specific elements are given by the matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Conjugating α_i by A_i gives rise to an action of $\mathbb{Z}_2$ which switches the order of the two eigenvalues of $\alpha_i(\mu_i)$ not equal to 1. The $\mathbb{Z}_2$ actions gives us discrete gluing parameters, and their overall effect on our count is to multiply by a factor of four. Thus, we see that the total number of isolated components in $R^*(M, SU(3))$ is $\frac{1}{4}(q_1^2 - 1)(q_2^2 - 1)$, and because the Casson invariant of the $(2, q)$ torus knot equals $\frac{1}{8}(q^2 - 1)$, we obtain the desired formula. $\square$

6 Cohomology calculations for *(p, q)*-torus knots

In this section, we present various cohomology results that are needed as input for the spectral flow computations in the next section, where we shall prove that the spectral flow to each of these $SU(3)$ representations is even. We choose a nice path of representations connecting the trivial representation to these $SU(3)$ representations and compute at which points the dimension of kernel of the odd signature operator with the boundary conditions from Definition 2.4 jumps.

Let K be the (p,q)-torus knot in S^3 and $X = S^3 \setminus \nu K$ its complement. We identify T (as in Section 2) with ∂X, such that the inclusion $j\colon T = \partial X \to X$ carries λ to a null-homologous loop in X. We orient X so that $-\partial X = T$, and we put a metric on X such that a collar of X is isometric to $[0,1] \times T$. The form dm on T extends to a closed 1-form on X generating the first cohomology $H^1(X;\mathbb{R})$, which we will continue to denote dm. In this section we will compute $\mathrm{Ker}(j^*)$ and $\mathrm{Im}(j^*)$, where $j^*\colon H^i(X;u(3)_\alpha) \to H^i(\partial X; u(3)_{j^*\alpha})$, $\alpha\colon \pi_1(X) \to S(U(2)\times U(1))$ is a representation, and $S(U(2)\times U(1))$ acts on $su(3)$ via the adjoint representation.

If we identify $S(U(2)\times U(1))$ with $U(2)$ via

$$\begin{pmatrix} tA & 0 \\ 0 & t^{-2} \end{pmatrix} \mapsto tA, \tag{6–1}$$

where $|t| = 1$ and $A \in SU(2)$, then $su(3)$ decomposes invariantly with respect to the adjoint action of $S(U(2)\times U(1))$ as

$$su(3) = u(2) \oplus \mathbb{C}^2,$$

where $tA \in U(2)$ acts on $u(2)$ via the adjoint representation and on $\mathbb{C}^2$ via multiplication by t^3A. If F is the covering from the $U(2)$ representation space of $\pi_1(X)$ to itself given by $F(\alpha)(w) := t^3A$ where $\alpha(w) = tA$ with $|t| = 1$ and $A \in SU(2)$, the twisted cohomology splits as

$$H^i(X; su(3)_\alpha) = H^i(X; u(2)_\alpha) \oplus H^i(X; \mathbb{C}^2_{F(\alpha)}),$$

where α acts by the adjoint representation on $u(2)$ and $F(\alpha)$ acts by the defining representation on $\mathbb{C}^2$. In this section, we concentrate on the case of $u(2)$ coefficients. There are analogous computations for the cohomology groups with $\mathbb{C}^2$ coefficients, see [5, Section 6.1] and [4, Section 3.1] for instance, but these computations are not needed here.

Proposition 6.1 *Let α be an $U(2)$ representation of $\pi_1(X)$, where $U(2)$ acts on $u(2)$*

via the adjoint representation. Then

$$\dim H^0(X;u(2)_\alpha) = \begin{cases} 4 & \text{if } \alpha \text{ is central,} \\ 2 & \text{if } \alpha \text{ is abelian, but not central,} \\ 1 & \text{otherwise.} \end{cases} \tag{6–2}$$

$$\dim H^1(X;u(2)_\alpha) = \begin{cases} 4 & \text{if } \alpha \text{ is abelian and } \alpha(x^p) \text{ is central,} \\ 2 & \text{otherwise.} \end{cases} \tag{6–3}$$

Proof The knot group $\pi_1(X)$ of the (p,q) torus knot $K \subset S^3$ admits the presentation

$$\pi_1(X) \cong \langle x, y \mid x^p = y^q \rangle.$$

Since every $U(2)$ matrix is diagonalizable, any representation $\alpha\colon \pi_1(X) \to U(2)$ can be conjugated so that

$$\alpha(x) = s \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}.$$

We will use the bar resolution to compute the cohomology. Let $\begin{pmatrix} ui & z \\ -\bar{z} & vi \end{pmatrix} \in u(2)$. Then

$$s\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}\begin{pmatrix} ui & z \\ -\bar{z} & vi \end{pmatrix}\left(s\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}\right)^{-1} = \begin{pmatrix} ui & 0 \\ 0 & vi \end{pmatrix} + \begin{pmatrix} a^2 & 0 \\ 0 & \bar{a}^2 \end{pmatrix}\begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$$

yields

$$\delta^0 \begin{pmatrix} ui & z \\ -\bar{z} & vi \end{pmatrix}(x) = \left(\mathrm{Id} - \begin{pmatrix} a^2 & 0 \\ 0 & \bar{a}^2 \end{pmatrix}\right)\begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}. \tag{6–4}$$

If α is central, then $\mathrm{Ker}(\delta^0) = u(2)$. If α is abelian and non-central, then $\alpha(y)$ is also diagonal, and

$$\mathrm{Ker}(\delta^0) = \mathrm{Ker}(\delta^0(\cdot)(x)) = \mathrm{Ker}(\delta^0(\cdot)(y))$$

is the 2–dimensional space of diagonal $u(2)$ matrices. If α is not abelian, then $\alpha(y)$ is not diagonal, and $\mathrm{Ker}(\delta^0(\cdot)(x))$ and $\mathrm{Ker}(\delta^0(\cdot)(y))$ are not equal. Then

$$\mathrm{Ker}(\delta^0) = \mathrm{Ker}(\delta^0(\cdot)(x)) \cap \mathrm{Ker}(\delta^0(\cdot)(y))$$

is 1–dimensional, because $\begin{pmatrix} ui & 0 \\ 0 & ui \end{pmatrix}$ commutes with conjugation. This shows (6–2).

Let ζ be a 1-cocycle. Then $\zeta(x) = X$ and $\zeta(y) = Y$ for $X, Y \in u(2)$ satisfying the equation

$$\sum_{i=0}^{p-1} x^i \cdot X = \sum_{i=0}^{q-1} y^i \cdot Y.$$

If α is central, the above equation simplifies to $pX = qY$ and the space of 1-cocycles is 4–dimensional. If α is non-central, we compute

$$\begin{aligned}\sum_{i=0}^{p-1} x^i \cdot X = \sum_{i=0}^{p-1} \begin{pmatrix} a^i & 0 \\ 0 & \bar{a}^i \end{pmatrix} \cdot X &= p \begin{pmatrix} ui & 0 \\ 0 & vi \end{pmatrix} + \sum_{i=0}^{p-1} \begin{pmatrix} a^{2i} & 0 \\ 0 & \bar{a}^{2i} \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \\ &= p \begin{pmatrix} ui & 0 \\ 0 & vi \end{pmatrix} + \begin{pmatrix} \frac{a^{2p}-1}{a^2-1} & 0 \\ 0 & \frac{\bar{a}^{2p}-1}{\bar{a}^2-1} \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}.\end{aligned} \tag{6–5}$$

If α is abelian and non-central, note that $\alpha(x)^p = \alpha(y)^q$ need not be central. A statement for y analogous to (6–5) then shows that the space of 1-cocycles is 4–dimensional if $\alpha(x)^p$ is non-central, and is 6–dimensional if $\alpha(x)^p$ is central. If α is irreducible, then $\alpha(x)^{2p} = \alpha(y)^{2q} = 1$. Then, just like for the 0-cocycles, $\mathrm{Ker}(\delta^1)$ does not contain all diagonal matrices of $u(2)$, but only those with equal entries. Therefore, in view of (6–5), the space of 1-cocycles is 5–dimensional for α irreducible. Since by (6–4) the space of 1-coboundaries is 0–dimensional for α central, 2–dimensional for α abelian and non-central, and 3–dimensional otherwise, (6–3) follows. □

Proposition 6.2 *Let α be an $U(2)$ representation of $\pi_1(T)$, where $U(2)$ acts on $u(2)$ via the adjoint representation. Then*

$$\dim H^0(T; u(2)_\alpha) = \begin{cases} 4 & \text{if } \alpha \text{ is central,} \\ 2 & \text{otherwise,} \end{cases} \tag{6–6}$$

$$\dim H^1(T; u(2)_\alpha) = \begin{cases} 8 & \text{if } \alpha \text{ is central,} \\ 4 & \text{otherwise.} \end{cases} \tag{6–7}$$

Proof The computation of (6–6) works just like the computation for (6–2), keeping in mind that all representations are abelian and we may assume that they are diagonal. For (6–7) note that a 1-cocycle ζ satisfies $\zeta(\lambda) - \mu \cdot \zeta(\lambda) = \zeta(\mu) - \lambda \cdot \zeta(\mu)$. For α non-central ζ is therefore uniquely determined up to coboundary (compare with (6–4)) by its values in the diagonal matrices. □

Together with the computations from Proposition 6.1 and Proposition 6.2 we can prove the following result. In the following, we decompose $u(2) = U \oplus W$ into diagonal and off-diagonal matrices, and further decompose $U = U' \oplus U''$, where

$$U = \left\{ \begin{pmatrix} ia & 0 \\ 0 & ib \end{pmatrix} \right\},\ U' = \left\{ \begin{pmatrix} ia & 0 \\ 0 & ia \end{pmatrix} \right\} \text{ and } U'' = \left\{ \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} \right\}.$$

Define $Q_{\alpha,\beta} = Q^{12}_{\alpha,\beta} \subseteq \Omega^0(T; W)$ to be the $u(2)$-analogue of the subspace described for $su(n)$ in (2–2) and (2–4), and recall the representation $\varphi_{\alpha,\beta}$ of $\pi_1(T)$ given just after Definition 2.1.

Theorem 6.3 *Suppose A is a $U(2)$ connection on X with $\mathrm{hol}(A) = \rho$ and $\rho|_T = \varphi_{\alpha,\beta}$. Then*

$$\mathscr{L}_A = \begin{cases} U \oplus Q_{\alpha,\beta} \oplus U\,dm \oplus Q_{\alpha,\beta}\,dm & \text{if } \rho \text{ is central,} \\ U \oplus U\,dm & \text{if } \rho \text{ is abelian, but not central,} \\ U' \oplus U\,(dm - pq\,d\ell) \oplus U''\,dm \wedge d\ell & \text{otherwise,} \end{cases} \tag{6–8}$$

and for $W_A := \mathrm{Ker}(H^1(X; u(2)_\rho) \to H^1(\partial X; u(2)_\rho))$

$$\dim(W_A) = \begin{cases} 2 & \text{if } \rho \text{ is non-central and } \rho(x^p) \text{ is central,} \\ 0 & \text{otherwise.} \end{cases} \tag{6–9}$$

Note that the non-central abelian representations with $\rho(x^p)$ central are twisted bifurcation points of the $SU(2)$ representation variety of the knot complement, that is, the 'T'-type intersections in the $SU(2)$ representation variety of the knot complement (see Figure 2), twisted in the sense of Definition 4.2.

Proof First observe that ρ is central if and only if its pull-back to $\pi_1(T)$ is central, because the meridian normally generates the fundamental group of the knot complement. Let us compute the limiting values of extended L^2-solutions. Notice that

$$\mathrm{Im}(H^1(X; u(2)_\rho) \to H^1(\partial X; u(2)_\rho))$$

is the differential of the restriction map $R(X, U(2)) \to R(T, U(2))$ for ρ non-central. For ρ central or ρ abelian with $\rho(x^p)$ non-central the computations are simple, and the result is obvious. If ρ is non-central and abelian with $\rho(x^p)$ central, we make use of the fact that $\mathrm{Im}(H^1(X; u(2)_\rho) \to H^1(\partial X; u(2)_\rho))$ is 2–dimensional and that it contains $U\,dm$. Let ρ be irreducible. We know that $\rho(\mu) = \varphi_{\alpha,\beta}(\mu)$ is diagonal. Then $\zeta(\mu) = M$ is diagonal and $\rho(x^p)$ is central. Therefore, $\zeta(\lambda) = -pqM$. Again, we make use of the fact that $\mathrm{Im}(H^1(X; u(2)_\rho) \to H^1(\partial X; u(2)_\rho))$ is 2–dimensional. Then we employ the de Rham theorem to prove (6–8).

Equation (6–9) follows directly from Proposition 6.1 and Proposition 6.2. □

7 The *SU(3)* Casson invariant of spliced sums

Suppose K_1 and K_2 are $(2, q_1)$ and $(2, q_2)$ torus knots with complements X_1 and X_2 in S^3, respectively, and let $M = X_1 \cup_T X_2$ denote their spliced sum. We shall relate the $SU(3)$ Casson invariant of M to the $SU(2)$ Casson invariants of $+1$ surgeries on K_1 and K_2, which are equal to the Casson knot invariants $\lambda'_{SU(2)}(K_1)$ and $\lambda'_{SU(2)}(K_2)$, using the approach of Taubes [16] to make the connection. This involves comparing various spectral flows, and in applying the results from the previous sections to X_2, we have to be careful with our parametrizations of the boundary: The parameters ℓ_1 and m_1 of ∂X_1 are identified with m_2 and ℓ_2. Let X_2 be with a metric and orientation as in Section 6. We orient X_1 such that $\partial X_1 = -T$ and place a metric on X_1, such that a collar of X_1 is isometric to $[-1,0] \times T$. It will be convenient to use the notation $\mathscr{P}^1 = \mathscr{P}^+$ and $\mathscr{P}^2 = \mathscr{P}^-$.

Let $B(t)$ be a path of $SU(3)$ connections on M with $B(0) = \Theta$ and $B(1)$ irreducible, such that $B(1)$ is reducible on either knot complement. By Lemma 3.1 and Theorem 2.10, it suffices to consider the spectral flow along a path of $S(U(2)\times U(1))$ and $S(U(1)\times U(2))$ connections on X_1 and X_2. Whenever convenient, identify $S(U(2) \times U(1))$ (and similarly $S(U(1) \times U(2))$) with $U(2)$ as in (6–1) with the induced action on $su(3) = u(2) \oplus \mathbb{C}^2$ as before. We can assume that each path is the composition of a path of $SU(2)$ connections with a path of twists of a fixed $SU(2)$ connection. The following definition makes this more precise.

Definition 7.1 Arrange paths $\tilde{A}_1(t)$ and $\tilde{A}_2(t)$ of $SU(2)$ connections, $t \in [0,\frac{1}{2}]$, as well as paths $A_1(t)$ and $A_2(t)$ of $SU(3)$ connections, $t \in [0,1]$, on the knot complement X_1 and X_2 respectively, satisfying

(1) $A_1(0) = \Theta$, $A_2(0) = \Theta$, $A_1(1) = B(1)|_{X_1}$, $A_2(1) = B(1)|_{X_2}$,

(2) $\tilde{A}_1(t)$ and $\tilde{A}_2(t)$ are paths of flat $SU(2)$ connections, and we denote by $A_1(t)$ and $A_2(t)$ the corresponding paths of $SU(2) \times \{1\}$ and $\{1\} \times SU(2)$ connections, and

(3) $\rho_1(t) := \operatorname{hol}(A_1(t))$ is a $\odot_1$-twist of $\operatorname{hol}(\tilde{A}_1(\frac{1}{2}))$ for $t \in [\frac{1}{2},1]$, and $\rho_2(t) := \operatorname{hol}(A_2(t))$ is a $\odot_2$-twist of $\operatorname{hol}(\tilde{A}_2(\frac{1}{2}))$ for $t \in [\frac{1}{2},1]$.

(4) $\tilde{\varrho}_1$ and $\tilde{\varrho}_2$ are paths in $\widetilde{\Lambda^2}$ with $A_i(t)|_T = a_{\varrho_i(t)}$ as in Definition 2.1, $\tilde{\varrho}_1(0) = \tilde{\varrho}_2(0)$ and $\tilde{\varrho}_1(1) = \tilde{\varrho}_2(1)$, where $\pi \circ \tilde{\varrho}_i = \varrho_i$ and $\pi\colon \widetilde{\Lambda^2} \to \Lambda^2 \cong \mathbb{R}^4$ the projection.

Figure 3 describes the situation in the case of a spliced sum of two trefoil complements. It shows their $SU(2)$ representation varieties immersed in the $SU(2)$ pillow case and the holonomy of $\tilde{A}_i(t)$, which is the untwisted part of the paths $A_i(t)$. The grey line is on the back of the pillowcase and the black line is on the front of the pillowcase. Let $\beta_{1,j}\colon \pi_1(X_1) \to SU(2)$ and $\beta_{2,j}\colon \pi_1(X_2) \to SU(2)$ be representations for $j = 1,\ldots,4$ such that

$$(\chi_{\theta_{1,j}} \odot_1 \beta_{1,j}) \cup (\chi_{\theta_{2,j}} \odot_2 \beta_{2,j})$$

are $SU(3)$ representations of $\pi_1(M)$. As in the proof of Theorem 5.1, we find four of the isolated $SU(3)$ representations of $\pi_1(M)$, and the others (there are 16 total) are obtained by applying the discrete gluing parameters.

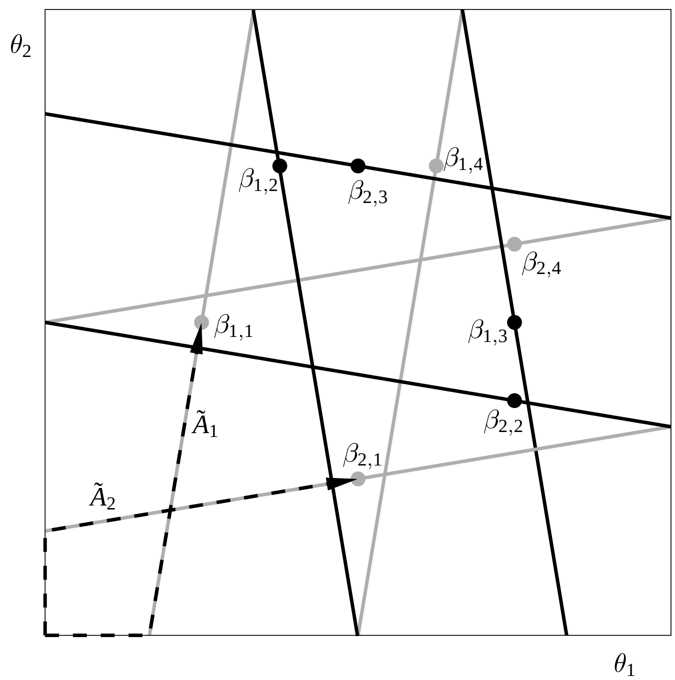

Figure 3: The $SU(2)$ representation varieties of two trefoils

By Theorem 2.10 we have

$$\begin{aligned}\mathrm{SF}(B(t)) = \ & \mathrm{SF}(A_1(t);\mathscr{P}^+_{\tilde{\varrho}_1(t)}) + \mathrm{SF}(A_2(t);\mathscr{P}^-_{\tilde{\varrho}_2(t)}) + \mathrm{SF}(\tilde{\varrho}_1(1-t) * \tilde{\varrho}_2(t)) \\ & + \tau_\mu(J\mathscr{L}_{X_1,0}, K^+_{\tilde{\varrho}_1(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,0}) - \tau_\mu(J\mathscr{L}_{X_1,1}, K^+_{\tilde{\varrho}_1(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,1}).\end{aligned}$$

In order to compute the above summands, we can break up the $su(3)$ spectral flow into $u(2)$ and $\mathbb{C}^2$ spectral flow. Note that the boundary conditions also respect the decomposition of $su(3)$. In particular, we will see in this section that the $\mathbb{C}^2$ spectral flow is even, and that the $u(2)$ spectral flow vanishes for $t \in [\frac{1}{2}, 1]$ and equals the $su(2)$ spectral flow along $\tilde{A}_1(t)$ or $\tilde{A}_2(t)$ for $t \in [0, \frac{1}{2}]$. Let us start with the easier case.

Proposition 7.2 *Let $A(t)$ be a path of $U(2)$ connections on X_i with $A(t)|_T = a_{\varrho(t)}$, $\varrho = \pi \circ \tilde{\varrho}$ and $\mathrm{hol}(A(t))$ acting on $\mathbb{C}^2$ via multiplication. Then $\mathrm{SF}_{\mathbb{C}^2}(A(t);\mathscr{P}^i_{\tilde{\varrho}(t)})$ is even.*

Proof Since $D_{A(t)}$ and $S_{a_{\varrho(t)}}$ are $\mathbb{C}$-linear, $\mathscr{P}^i_{\tilde{\varrho}(t)} \cap L^2(\Omega^{0+1+2}(T;\mathbb{C}^2))$ is a vector space over $\mathbb{C}$ and the eigenspaces of $D_{A(t)}$ with boundary conditions $\mathscr{P}^i_{\tilde{\varrho}(t)} \cap L^2(\Omega^{0+1+2}(T;\mathbb{C}^2))$ are complex subspaces. Therefore, the eigenvectors come in pairs and the (real) spectral flow is even as claimed. □

We will need the following lemma for various computations.

Lemma 7.3 *Let $A(t)$ be any path of irreducible $U(2)$ connections on X_i with $A(t)|_T = a_{\varrho(t)}$ and $\varrho = \pi \circ \tilde{\varrho}$. Then*

$$\mathrm{SF}_{u(2)}(A(t); \mathscr{P}^i_{\tilde{\varrho}(t)}) = 0$$

Proof Consider the case $i = 1$. The computation of the limiting values of extended L^2-solutions in Theorem 6.3 and the definition of $\hat{\mathscr{L}}$ in Definition 2.4 show that for $\mathrm{hol}(A) = a_{\alpha,\beta}$, θ arbitrary, and $u(2)$ coefficients,

$$\Lambda^\infty_A \cap \mathscr{P}^+_{\alpha,\beta,\theta} = \mathscr{L}_A \cap \hat{\mathscr{L}}^+ = U''\, dm \wedge d\ell,$$

and hence by [12, Lemma 8.10]

$$\dim \mathrm{Ker}(D_A; \mathscr{P}^+_{\alpha,\beta,\theta}) = \dim(\Lambda^\infty_A \cap \mathscr{P}^+_{\alpha,\beta,\theta}) = 1.$$

Therefore, there is no $u(2)$ spectral flow along a path of irreducibles. A similar computation for $i = 2$ completes the proof. □

The $SU(3)$ Casson invariant of M is a signed count of irreducible $SU(3)$ representations of $\pi_1(M)$. By Theorem 2.10, this sign is determined by the $su(3)$ spectral flow on X_i, $i = 1, 2$, to these representations. The following proposition motivates the appearance of the $SU(2)$ Casson invariant: this $su(3)$ spectral flow is equal to the $su(2)$ spectral flow to certain irreducible $SU(2)$ connections on X_i. It turns out that there is a fixed number of such irreducible $SU(2)$ connections associated to each irreducible $SU(2)$ representation, the signed count of which is the $SU(2)$ Casson invariant.

Proposition 7.4 *For the path $A_i(t)$ given in Definition 7.1, we have*

(7–1) $$\mathrm{SF}_{u(2)}(A_i(t); \mathscr{P}^i_{\tilde{\varrho}_i(t)}) = \mathrm{SF}_{su(2)}(A_i(t); \mathscr{P}^i_{\tilde{\varrho}_i(t)}), \quad t \in [0, \tfrac{1}{2}],$$

(7–2) $$\mathrm{SF}_{u(2)}(A_i(t); \mathscr{P}^i_{\tilde{\varrho}_i(t)}) = 0, \quad t \in [\tfrac{1}{2}, 1].$$

Proof By Theorem 6.3 we get for $\mathrm{hol}(A) = a_{\alpha,\beta}$ and θ arbitrary

$$\mathrm{Ker}_{u(2)}(D_A; \mathscr{P}^+_{\alpha,\beta,\theta}) = U' \oplus \mathrm{Ker}_{su(2)}(D_A; \mathscr{P}^+_{\alpha,\beta,\theta})$$

and $$\mathrm{Ker}_{u(2)}(D_A; \mathscr{P}^-_{\alpha,\beta,\theta}) = \mathrm{Ker}_{su(2)}(D_A; \mathscr{P}^-_{\alpha,\beta,\theta}) \oplus U''\, dm \wedge d\ell.$$

Since $su(2)$ eigenfunctions are also $u(2)$ eigenfunctions, we get (7–1). Lemma 7.3 and the Remark in Section 5 yield (7–2). □

Let X_1^+ and X_2^+ be $+1$ surgery on the corresponding knots. Let $S_i = X_i^+ \backslash X_i$, which is a solid torus, whose $SU(2)$ representation variety maps into the pillow case as the diagonal. A simple computation analogous to Theorem 6.3 gives the limiting values of extended L^2-solutions $\mathscr{L}_{S_i}$ with $su(n)$ coefficients for S_i keeping in mind the parametrization induced by surgery.

Lemma 7.5 *Let A be an $SU(n)$ connection on S_i with $\mathrm{hol}(A) = \rho$ and $\rho|_T = \varphi_{\alpha,\beta}$. Decompose $su(n) = U_n \oplus W_n$ into diagonal and off-diagonal matrices as before and let $Q_{\alpha,\beta}$ be as defined in equation (2–4). Then*

$$\mathscr{L}_{S_i,\alpha,\beta} = \begin{cases} U_n \oplus Q_{\alpha,\beta} \oplus U_n(dm + d\ell) \oplus Q_{\alpha,\beta}(dm + d\ell) & \text{if } \rho \text{ is central,} \\ U_n \oplus U_n\,(dm + d\ell) & \text{otherwise.} \end{cases}$$

By Lemma 7.3 we can elongate $\tilde{A}_i(t)$, $t \in [0, \frac{1}{2}]$, by a path of irreducible $SU(2)$ connections to a path $\tilde{A}_i(t)$ of flat connections on X_i such that $\tilde{A}_i(1)$ can be extended flatly to $\tilde{A}_i'(t)$ on X_i^+. We assume that $a_{\sigma_i(t)} := A_i(t)|_T$, $\pi \circ \tilde{\sigma}_i(t) = \sigma_i(t)$ for some path $\tilde{\sigma}_i$ which agrees with $\tilde{\varrho}_i$ for $t \in [0, \frac{1}{2}]$. Working modulo 2, we apply Theorem 2.10, Lemma 7.3, Proposition 7.2, Proposition 7.4, and Proposition 2.9 to see that

$$\begin{aligned} \mathrm{SF}_{su(3)}(B(t)) \equiv{}& \mathrm{SF}_{su(2)}(\tilde{A}_1(t); \mathscr{P}^+_{\tilde{\sigma}_1(t)}) + \mathrm{SF}_{su(2)}(\tilde{A}_2(t); \mathscr{P}^-_{\tilde{\sigma}_2(t)}) \\ &+ \tau_\mu(J\mathscr{L}_{X_1,\varrho_1(0)}, K^+_{\tilde{\varrho}_1(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\varrho_1(0)}) - \tau_\mu(J\mathscr{L}_{X_1,\varrho_1(1)}, K^+_{\tilde{\varrho}_1(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\varrho_1(1)}) \\ &- \tau_\mu(J\mathscr{L}_{X_1,\sigma_1(0)}, K^+_{\tilde{\sigma}_1(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{S_2,\sigma_1(0)}) + \tau_\mu(J\mathscr{L}_{X_1,\sigma_1(1)}, K^+_{\tilde{\sigma}_1(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{S_2,\sigma_1(1)}) \\ &- \tau_\mu(J\mathscr{L}_{S_1,\sigma_2(0)}, K^+_{\tilde{\sigma}_2(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\sigma_2(0)}) + \tau_\mu(J\mathscr{L}_{S_1,\sigma_2(1)}, K^+_{\tilde{\sigma}_2(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\sigma_2(1)}). \end{aligned}$$

Note that the Maslov triple indices in the last two lines are with respect to $su(2)$ coefficients, while the first two Maslov triple indices are with respect to $su(3)$ coefficients. It remains to show that these Maslov triple indices add up to an even number.

Recall that in general S_a and D_A preserve the decomposition $su(n) = U_n \oplus W_n$ into diagonal and off-diagonal parts and are complex linear on the forms with values in the off-diagonal matrices. Therefore, we only need to consider the triple Maslov indices on the forms with values in the diagonal $su(n)$ matrices, because the contribution from the off-diagonal $su(n)$ matrices is always even. Furthermore, the remaining Lagrangians are direct sums of Lagrangian subspaces of $L^2(\Omega^{0+2}(T; U_n))$ and $L^2(\Omega^1(T; U_n))$. As before, we identify $su(3)$ with $u(2) \oplus \mathbb{C}^2$ and also U_3 with U in order to apply

Theorem 6.3 to see that, modulo 2, we have

$$\tau_\mu(J\mathscr{L}_{X_1,\varrho_1(0)}, K^+_{\tilde{\varrho}_1(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\varrho_1(0)}) \equiv \tau_\mu(U\,dm \wedge d\ell, U\,dm \wedge d\ell, U) + \tau_\mu(U\,dm, U\,dm, U\,dm), \tag{7–3}$$

$$\begin{aligned}\tau_\mu(J\mathscr{L}_{X_1,\varrho_1(1)}, K^+_{\tilde{\varrho}_1(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\varrho_1(1)}) \equiv{}& \tau_\mu(U'\,dm \wedge d\ell, U'\,dm \wedge d\ell, U') \\ &+ \tau_\mu(U'', U''\,dm \wedge d\ell, U''\,dm \wedge d\ell) \\ &+ \tau_\mu(U\,(dm + pq\,d\ell), U\,dm, U\,(dm - pq\,d\ell)).\end{aligned} \tag{7–4}$$

Clearly the Maslov triple indices on the right side of (7–3) and the first two on the right side of (7–4) vanish by Lemma 2.5. For the third Maslov triple index on the right side of (7–4), note that U is 2–dimensional. Therefore, (7–3) and (7–4) are congruent to 0 mod 2.

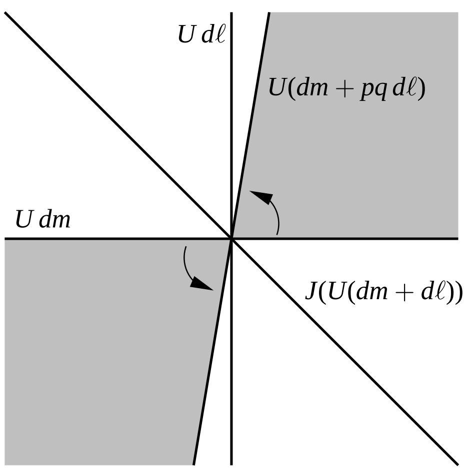

Figure 4: Path for (7–7)

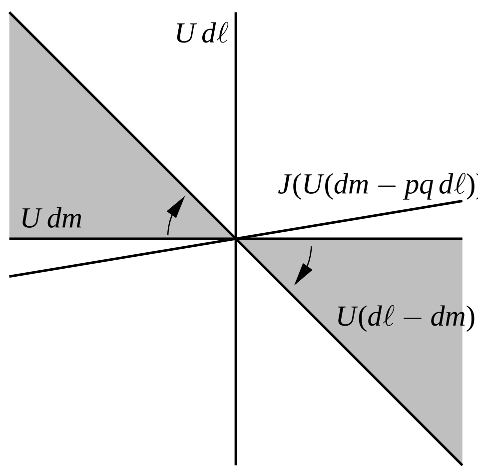

Figure 5: Path for (7–8)

For the Maslov triple indices concerning the $su(2)$ coefficients, we let $U = U_2$ and see that, modulo two, we have

$$\begin{aligned}\tau_\mu(J\mathscr{L}_{X_1,\sigma_1(0)}, K^+_{\tilde\sigma_1(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{S_2,\sigma_1(0)}) &\equiv \tau_\mu(U\,dm \wedge d\ell, U\,dm \wedge d\ell, U)\\ &\quad + \tau_\mu(U\,dm, U\,dm, U\,(dm + d\ell)),\end{aligned} \tag{7–5}$$

$$\begin{aligned}\tau_\mu(J\mathscr{L}_{S_1,\sigma_2(0)}, K^+_{\tilde\sigma_2(0)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\sigma_2(0)}) &\equiv \tau_\mu(U\,dm \wedge d\ell, U\,dm \wedge d\ell, U)\\ &\quad + \tau_\mu(U\,(d\ell - dm), U\,dm, U\,dm),\end{aligned} \tag{7–6}$$

$$\begin{aligned}\tau_\mu(J\mathscr{L}_{X_1,\sigma_1(1)}, K^+_{\tilde\sigma_1(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{S_2,\sigma_1(1)}) &\equiv \tau_\mu(U, U\,dm \wedge d\ell, U\,dm \wedge d\ell)\\ &\quad + \tau_\mu(U\,(dm + pq\,d\ell), U\,dm, U\,(dm + d\ell)),\end{aligned} \tag{7–7}$$

$$\begin{aligned}\tau_\mu(J\mathscr{L}_{S_1,\sigma_2(1)}, K^+_{\tilde\sigma_2(1)} \oplus \hat{\mathscr{L}}^+, \mathscr{L}_{X_2,\sigma_2(1)}) &\equiv\\ &\tau_\mu(U\,dm \wedge d\ell, U\,dm \wedge d\ell, U\,dm \wedge d\ell)\\ &\quad + \tau_\mu(U\,(d\ell - dm), U\,dm, U\,(dm - pq\,d\ell)).\end{aligned} \tag{7–8}$$

Again, the Maslov triple indices on the right side of (7–5) and (7–6) vanish by Lemma 2.5. One can see that the Maslov triple indices on the right side of equations (7–7) and (7–8) vanish as follows. Choose the shortest path from $U\,dm$ to $U\,(dm+pq\,d\ell)$ by a rotation as indicated in Figure 4 and notice that this path intersects neither $U\,d\ell = J(U\,dm)$ nor $J(U\,(dm+d\ell))$. Similarly Figure 5 describes the situation for a path from $U\,dm$ to $U\,(d\ell - dm)$ by a rotation, which intersects neither $J(U\,dm)$ nor $J(U\,(dm - pq\,d\ell))$. In summary, all Maslov triple indices in our formula are even as claimed.

Recall that every contribution to the $SU(2)$ Casson invariant is positive. Then we get the following result directly from Theorem 5.1.

Theorem 7.6 *Suppose K_1 and K_2 are torus knots of type $(2,q_1)$ and $(2,q_2)$, respectively, and M is their spliced sum. Then*

$$\lambda_{SU(3)}(M) = 16\,\lambda'_{SU(2)}(K_1)\,\lambda'_{SU(2)}(K_2),$$

where $\lambda'_{SU(2)}(K)$ is the $SU(2)$ Casson knot invariant normalized to be 1 for the trefoil.

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