# INVARIANTS OF FIBRED KNOTS FROM MODULI

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In this paper, invariants of fibred knots are described in terms of unitary representation spaces of fundamental groups of Seifert surfaces. Because these representation spaces admit an interpretation as moduli spaces of parabolic bundles [10], the results of [3, 5, 6] can be used to calculate the invariants. The main result here is Theorem 3. The proof of this theorem involves the theory of parabolic bundles in an essential way, but for the sake of clarity we have suppressed their role. These issues will be addressed in [4].

The knot invariants, denoted $\mu_\alpha(\mathbf{K})$ or simply $\mu_\alpha$ when the knot is understood, are defined for fibred knots $\mathbf{K}$ in a rational homology sphere $\mathbf{M}$ and for $\alpha \in \mathbf{SU}_n$ a regular value of the map $\widetilde{\Psi}$ described in Definition 1. They are a generalization of Frohman's invariants, $\mu_\omega$, which are invariants of fibred knots defined for $\omega$ a regular value of $\widetilde{\Psi}$ in the center of $\mathbf{SU}_n$ [7]. These have already been extended to arbitrary knots in rational homology spheres in the cases

(1) where $\omega$ is a regular value of $\widetilde{\Psi}$ in the center of $\mathbf{SU}_n$ in [8], and
(2) where $\omega$ is any element in the center of $\mathbf{SU}_n$ in [9].

The invariants presented in this paper form a continuous family of invariants parameterized by conjugacy classes of regular values $\alpha$ of $\widetilde{\Psi}$. The most interesting aspect of this approach is the behavior of $\mu_\alpha$ as $\alpha$ is allowed to vary in the set

$$W \stackrel{\text{def}}{=} \mathbf{SU}_n/\mathrm{Ad}.$$

The critical values of $\Psi$ form a union of hyperplanes, giving $W$ a natural chamber structure. Theorem 3 describes the behavior of these invariants in terms of this chamber structure. Part (1) of Theorem 3 states that $\mu_\alpha = \mu'_\alpha$ for $\alpha$ and $\alpha'$ in the same chamber, part (2) compares $\mu_\alpha$ and $\mu_\beta$ for $\alpha$ and $\beta$ in adjacent chambers, and part (3) relates $\mu_\alpha$ and $\mu_\omega$ for $\alpha$ in the interior and $\omega$ on the boundary of $W$. The interest in these invariants lies in their ability to detect certain

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irreducible representations of the knot group: if $\mu_\alpha \neq 0$ then there exists an irreducible unitary representation of the fundamental group of the knot complement with holonomy along the longitude conjugate to $\alpha$.

To start, we fix some notation. Given an arbitrary group $\pi$ and a compact Lie group $\mathbf{G}$, let

$$\widetilde{\mathbf{R}}(\pi, \mathbf{G}) = \operatorname{Hom}(\pi, \mathbf{G})$$

denote the space of homomorphisms of $\pi$ into $\mathbf{G}$. The group $\mathbf{G}$ acts on $\widetilde{\mathbf{R}}(\pi, \mathbf{G})$ by conjugation (denoted by Ad) and we get the quotient

$$\mathbf{R}(\pi, \mathbf{G}) = \widetilde{\mathbf{R}}(\pi, \mathbf{G})/\mathrm{Ad}$$

which is called the space of representations of $\pi$ in $\mathbf{G}$.

To simplify notation, set $\pi = \pi_1 F$, the fundamental group of a closed Riemann surface $F$ of genus $g$ and $\pi^* = \pi_1 F^*$, the fundamental group of $F^* = F \setminus D^2(p)$, where $D^2(p)$ is a small disk centered at $p \in F$. Then $\pi$ admits the presentation

$$\pi = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle, \tag{1}$$

and $\pi^*$ is simply the free group on the $2g$ generators $a_1, b_1, \dots, a_g, b_g$. This last fact gives an identification

$$\begin{aligned} \widetilde{\mathbf{R}}(\pi^*, \mathbf{G}) \ &\cong \overbrace{\mathbf{G} \times \cdots \times \mathbf{G}}^{2g}, \\ \rho \ &\mapsto (A_1, B_1, \dots, A_g, B_g) \end{aligned}$$

where $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$. Setting $\partial = \prod_{i=1}^{g} [a_i, b_i] \in \pi^*$ we define maps $\widetilde{\Psi}$ and $\Psi$ on $\widetilde{\mathbf{R}}$ and $\mathbf{R}$ by evaluation on $\partial$.

**Definition 1.** Let $\widetilde{\Psi} : \widetilde{\mathbf{R}}(\pi^*, \mathbf{G}) \longrightarrow \mathbf{G}$ be defined using the above identification and setting $\widetilde{\Psi}(A_1, B_1, \dots, A_g, B_g) = \prod_{i=1}^{g}[A_i, B_i]$. Since $\widetilde{\Psi}(g \cdot \rho) = g\widetilde{\Psi}(\rho)g^{-1}$, we can define $\Psi : \mathbf{R}(\pi^*, \mathbf{G}) \longrightarrow \mathbf{G}/\mathrm{Ad}$ and the following square commutes:

$$\begin{array}{ccc} \widetilde{\mathbf{R}}(\pi^*, \mathbf{G}) & \xrightarrow{\widetilde{\Psi}} & \mathbf{G} \\ {\scriptstyle \mathrm{Ad}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}} \\ \mathbf{R}(\pi^*, \mathbf{G}) & \xrightarrow{\Psi} & \mathbf{G}/\mathrm{Ad}. \end{array} \tag{2}$$

By definition, $\mathrm{im}(\widetilde{\Psi}) \subset [\mathbf{G},\mathbf{G}]$, the commutator subgroup of $\mathbf{G}$. The surjection $\pi^* \to \pi$ induces an inclusion $\mathbf{R}(\pi,\mathbf{G}) \hookrightarrow \mathbf{R}(\pi^*,\mathbf{G})$ defined by pullback, and it is clear that $\rho \in \mathrm{im}(i) \Longleftrightarrow \Psi(\rho) = I$.

For the remainder of the paper, we set $\mathbf{G} = \mathbf{U}_n$, the group of unitary $n \times n$ complex matrices. Recall that $\mathbf{SU}_n = [\mathbf{U}_n, \mathbf{U}_n]$, and denote by $\mathbf{Z}_n$ the center of $\mathbf{SU}_n$ and by $\mathbf{PU}_n$ the quotient $\mathbf{SU}_n/\mathbf{Z}_n$.

Suppose now that $\mathbf{K}$ is a fibred knot in a homology sphere $\mathbf{M}$ with spanning surface $F^*$. Let

$$\varphi \,:\, F^* \xrightarrow{\approx} F^*$$

be the monodromy of the fibration, so the knot complement is given by

$$\mathbf{M}_{\mathbf{K}} = F^* \times I/\sim,$$

where $(x,0) \sim (\varphi(x),1)$.

In [7], Frohman defines invariants of fibred knots by considering the Lefschetz number of the monodromy action on certain smooth submanifolds of $\mathbf{R}(\pi^*,\mathbf{U}_n)$, namely $\mathbf{R}_\omega$ (defined below), where $\omega \in \mathbf{Z}_n$ is a regular value of $\Psi$. This is the case if and only if $\omega = e^{2\pi i k/n} I$, where $k$ and $n$ are relatively prime.

We now describe an extension of these invariants. For any $\alpha \in \mathbf{SU}_n$, denote by $C(\alpha)$ the orbit $\mathrm{Ad}\cdot\alpha$ of $\alpha$ under conjugation and set $\widetilde{\mathbf{R}}_\alpha = \widetilde{\Psi}^{-1}(C(\alpha))$. A standard result shows that the critical points of $\widetilde{\Psi}$ are precisely the reducible representations (see [1] or [8]). In particular, if $\alpha$ is a regular value of $\widetilde{\Psi}$, then $\widetilde{\Psi}^{-1}(C(\alpha))$ is smooth and consists entirely of irreducible representations.[1]

Define

$$\mathbf{R}_\alpha = \widetilde{\mathbf{R}}_\alpha/\mathrm{Ad} = \Psi^{-1}(C(\alpha)).$$

Since the adjoint action is a free $\mathbf{PU}_n$ action on the irreducible representations, it follows that for $\alpha$ a regular value, $\mathbf{R}_\alpha$ is a compact oriented manifold of dimension

$$\dim \mathbf{R}_\alpha = (2g-2)n^2 + 2 + \dim C(\alpha).$$

The invariants $\mu_\alpha$ are defined as follows.

**Definition 2.** (1) If $P$ is a compact oriented manifold and $f : P \to P$, then the Lefschetz number of $f$, denoted $\Lambda(f,P)$, is the algebraic

[1] It is enough to assume that $\widetilde{\Psi}$ is transverse to $C(\alpha)$, but a simple calculation shows that if $\widetilde{\Psi}$ is transverse to $C(\alpha)$, then $\alpha$ is indeed a regular value.

intersection number of the graph of $f$ with the diagonal $\triangle(P)$ in $P \times P$. Further, the Lefschetz polynomial is defined to be

$$L(f,P)(t) = \sum_{0 \le n} (-1)^n \mathrm{Tr}(f^* : \mathrm{H}^n(P,\mathbb{Q}) \to \mathrm{H}^n(P,\mathbb{Q}))t^n.$$

(2) Let $\mu_\alpha(\mathbf{K})$ denote the Lefschetz fixed point number of the map on the representation variety induced by the monodromy of the knot, i.e.

$$\mu_\alpha(\mathbf{K}) = \Lambda(\varphi^*, \mathbf{R}_\alpha).$$

(3) Let $M_\alpha(\mathbf{K},t)$ denote the Lefschetz polynomial of the map on the cohomology of the representation variety induced by the monodromy of the knot, i.e.

$$M_\alpha(\mathbf{K},t) = L(\varphi^*, \mathbf{R}_\alpha).$$

The Lefschetz fixed point theorem implies that the Lefschetz number equals the Lefschetz polynomial evaluated at $t=1$, thus $\mu_\alpha = M_\alpha(1)$. Of course, if $\alpha$ is conjugate to $\alpha'$, then $\mu_\alpha = \mu_{\alpha'}$. In the other cases, we would like to compare the invariants $\mu_\alpha$ and $\mu_\beta$. One technique for doing this is to let $\alpha_t$ be a path connecting $C(\alpha)$ and $C(\beta)$ in $W$ and to consider the map $\Psi$ restricted to the preimage of this path. By choosing the path carefully (e.g. so it is transverse to the critical values of $\Psi$), Morse theory for $\Psi$, in the sense of Bott, constructs a cobordism between $\mathbf{R}_\alpha$ to $\mathbf{R}_\beta$. The invariants $\mu_\alpha$ and $\mu_\beta$ can then be studied in terms of the change in the cohomology under this cobordism. This is the rough idea behind our approach.

To begin, identify the critical values of $\Psi$ with a union of hyperplanes in $W$ as follows. Notice that $W$ is an $(n-1)$-simplex. (Choose a positive Weyl chamber in $\mathbf{su}_n$). Use the following natural (but discontinuous!) coordinates on $W$. Since any $\alpha \in \mathbf{SU}_n$ is conjugate to a matrix of the form

$$\exp(diag(\alpha_1,\dots,\alpha_n)) = \begin{pmatrix} e^{2\pi i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i \alpha_n} \end{pmatrix}$$

where $0 \le \alpha_1 \le \cdots \le \alpha_n < 1$ and $\sum_{i=1}^n \alpha_i$ is an integer, it follows that we can use $(\alpha_1,\dots,\alpha_n)$ to give Ad-invariant coordinates to $\mathbf{SU}_n$. The resulting coordinates on $W$ are discontinuous precisely when $\alpha_1 = 0$, which can be seen by considering the coordinates assigned to the path $\alpha_t = \exp(diag(t, \alpha_2 - \frac{t}{n-1}, \dots, \alpha_n - \frac{t}{n-1}))$ for $t \in (-\epsilon, \epsilon)$.

To deal with these discontinuities, decompose $W = \bigcup_{k=0}^{n-1} W_k$, where

$$W_k = \{(\alpha_1, \dots \alpha_n) \in W \mid \sum_{i=1}^{n} \alpha_i = k\},$$

and notice that the coordinates are continuous along each $W_k$. The hyperplanes $\alpha_1 = 0$ lie in $\partial W_k$ and are called *bad* hyperplanes. Of course, $W$ can be reassembled by identifying these bad hyperplanes to the hyperplanes $\alpha_n = 1$, which lie in $\partial \overline{W_{k+1}}$, in the more or less obvious manner.

Suppose now that $\rho : \pi_1(F^*) \to \mathbf{U}_n$ is a reducible representation. Then up to conjugacy, we have $\mathrm{im}(\rho) \subset \mathbf{U}_{n_1} \times \mathbf{U}_{n_2}$. Because $\gamma = \widetilde{\Psi}(\rho)$ is contained in the commutator subgroup of $\mathrm{im}(\rho)$, which is just $\mathbf{SU}_{n_1} \times \mathbf{SU}_{n_2}$, $\gamma$ is conjugate to a matrix in block form

$$\begin{pmatrix} \gamma^1 & 0 \\ 0 & \gamma^2 \end{pmatrix}$$

where $\gamma^i \in \mathbf{SU}_{n_i}$ for $i = 1, 2$. Writing $\gamma = \exp(diag(\gamma_1, \dots, \gamma_n))$, we see that $\gamma = \widetilde{\Psi}(\rho)$ for a reducible $\rho$ if and only if there is a proper subcollection $0 \le \gamma_{\sigma(1)} \le \cdots \le \gamma_{\sigma(n_1)} < 1$ with $\sum_{j=1}^{n_1} \gamma_{\sigma(j)}$ an integer.

This shows that the collection of critical values of $\Psi$ (which are the projection under Ad of the critical values of $\widetilde{\Psi}$) are given by a union of hyperplanes $\bigcup_\xi H_\xi$ in $W$, some of which are *good* (i.e. not bad), others of which are bad. We call the connected components of $W \setminus \bigcup_\xi H_\xi$ *chambers.*

Suppose $\alpha, \beta \in W_k$ are regular values of $\Psi$ in adjacent chambers and are separated by a good hyperplane $H$. Choose $\gamma \in H$, a generic point lying on the hyperplane separating $\alpha$ and $\beta$ (genericity means that $\gamma$ lies on no other hyperplane). Denote by $\Sigma_\gamma$ the reducible representations of $\mathbf{R}_\gamma$. Then with a choice of a complex structure $J$ on $F$, the theorem of Mehta and Seshadri [10] provides an identification of each of these representation spaces with a corresponding moduli space of semistable parabolic bundles. Roughly speaking, the parabolic structure is determined by the eigenspaces and eigenvalues of the matrices ($\alpha, \beta$ or $\gamma$). In this way, each of the representation spaces $\mathbf{R}_\alpha, \mathbf{R}_\beta$, and $\mathbf{R}_\gamma$ inherits the structure of a normal, projective variety. Furthermore, since $\gamma$ lies on only one hyperplane, there is only one way for a representation in $\mathbf{R}_\gamma$ to reduce, implying that $\Sigma_\gamma$ is smooth and is in fact the product of the lower dimensional representation spaces $\mathbf{R}_{\gamma^1} \times \mathbf{R}_{\gamma^2}$. The next theorem follows from Theorem 3.1 of [5] (cf. Theorem 5.3 of [6] in the case of a bad hyperplane).

**Theorem 1.** *Choose $\alpha, \beta, \gamma \in W$ as above and $J$ a complex structure on $F$. Then the representation spaces $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$ are related by a special birational transformation (like a flip in Mori theory), i.e. there are projective maps $\Phi_\alpha$ and $\Phi_\beta$*

$$\begin{array}{ccc} \mathbf{R}_\alpha & & \mathbf{R}_\beta \\ & \phi_\alpha \searrow \; \swarrow \phi_\beta & \\ & \mathbf{R}_\gamma & \end{array}$$

*(depending a priori on $J$) with the properties that*

(1) *along $\mathbf{R}_\gamma \setminus \Sigma_\gamma$, $\phi_\alpha$ and $\phi_\beta$ are isomorphisms,*
(2) *along $\Sigma_\gamma$, $\phi_\alpha$ and $\phi_\beta$ are $\mathbb{CP}^a$ and $\mathbb{CP}^b$ bundles, respectively, and*
(3) *$a + b + 1 = \mathrm{codim}_{\mathbb{C}} \Sigma_\gamma$.*

We can also compare the spaces $\mathbf{R}_\alpha$ and $\mathbf{R}_\omega$, where $\alpha$ and $\omega$ are regular values of $\Psi$ in the interior and on the boundary of $W$, respectively. By the previous theorem, we can assume that $\alpha$ and $\omega$ are not separated by any hyperplanes. The following is a restatement of Proposition 3.4 of [5].

**Proposition 2.** *There is a natural projection $\psi : \mathbf{R}_\alpha \longrightarrow \mathbf{R}_\omega$ which is a fiber bundle with fiber a flag variety $\mathcal{F}$.*

Theorem 1 and Proposition 2 can now be used to describe the behavior of the invariants $\mu_\alpha$ as $\alpha$ is allowed to vary within $W$.

**Theorem 3.** *Suppose $\mathbf{K}$ is a knot in a homology sphere. The invariants $\mu_\alpha(\mathbf{K})$ depend on $\alpha \in W$ in the following way.*

(1) *If $\alpha$ and $\alpha'$ are regular values of $\Psi$ in the interior of $W$ contained in the same chamber, then*
$$\mu_\alpha = \mu_{\alpha'}.$$
(2) *If $\alpha, \beta$ and $\gamma$ are chosen as in Theorem 1, then*
$$\mu_\alpha - \mu_\beta = (\chi(\mathbb{CP}^a) - \chi(\mathbb{CP}^b))\mu_{\gamma^1}\mu_{\gamma^2}.$$
(3) *If $\alpha$ and $\omega$ are chosen as in Proposition 2, then*
$$\mu_\alpha = \chi(\mathcal{F})\mu_\omega.$$

*Sketch of proof.* To prove part (1), notice that $\mathbf{R}_\alpha$ is diffeomorphic to $\mathbf{R}_{\alpha'}$. This follows because $\alpha$ and $\alpha'$ are connected by a path $\alpha_t$ which misses the hyperplanes, hence $\Psi^{-1}(\alpha_t)$ is a product. The map $\varphi^*$ acts on this product and preserves each fiber, hence the family of maps $\varphi_t^*$ on $\mathbf{R}_{\alpha_t}$ describes, after pullback to $\mathbf{R}_\alpha$, an isotopy from $\varphi_0^*$ to the pullback of $\varphi_1^*$. Part (1) follows since Lefschetz numbers are invariants of the isotopy class of a map.

For part (2), notice that a direct consequence of Theorem 1 is the identity

$$(*) \qquad \mathrm{P}_t(\mathbf{R}_\alpha) = \mathrm{P}_t(\mathbf{R}_\beta) + (\mathrm{P}_t(\mathbb{CP}^a) - \mathrm{P}_t(\mathbb{CP}^b))\mathrm{P}_t(\Sigma_\gamma),$$

where $\mathrm{P}_t$ denotes the Poincaré polynomials taken with $\mathbb{Z}$ coefficients (cf. Corollary 3.2 of [5]). We would like to see that this formula also holds on the level of the Lefschetz polynomials, i.e. that

$$(**) \qquad M_\alpha(t) - M_\beta(t) = (\mathrm{P}_t(\mathbb{CP}^a) - \mathrm{P}_t(\mathbb{CP}^b))M_{\gamma^1}(t)M_{\gamma^2}(t).$$

The direct route is to give a general algorithm for the computation of $M_\alpha(t)$ from which one can conclude $(**)$.

To explain this algorithm, we describe first the Atiyah-Bott-Nitsure method to calculate $\mathrm{P}_t(\mathbf{R}_\alpha)$. Consider the case $\omega \in \mathbf{Z}_n$. By [2], we see that $\mathrm{P}_t(\mathbf{R}_\omega)$ is described as a difference of two terms, both of which are infinite series in $t$, the first being the cohomology of the classifying space of the gauge group (which is determined by the rank and genus, cf. Theorem 2.15 of [2]) and the second admitting an expression as a power series whose coefficients are polynomial functions in $(1+t)^{2g}$, the Poincaré polynomial of the Jacobian of the surface (see formulas (11.1) – (11.3) of [2]). This formula results from an equivariantly perfect stratification on the space of holomorphic structures on a fixed topological bundle over the Riemann surface $(F, J)$.

Frohman adapted these ideas in order to compute $\mu_\omega$ in [7]. He expressed the Lefschetz polynomial $M_\omega(t)$ as a difference of two Lefschetz traces (which are formal power series in $t$), the first of which is determined by the rank and genus and the second of which admits an expression as a power series in $t$ whose coefficients are polynomial functions in $c(t) = L(F^*, \varphi^*)$, the Lefschetz polynomial of the monodromy acting on the Jacobian. Of course, $c(t)$ is just the Alexander polynomial and evaluating $M_\omega(t)$ at $t = 1$ gives a formula for the Lefschetz number $\mu_\omega$ in terms of the derivatives of the Alexander polynomial of the knot evaluated at $t = 1$ (see Theorems 1.6 and 3.14 of [7]).

The Atiyah-Bott procedure was extended to parabolic bundles in [11] and used to give explicit computations of $\mathrm{P}_t(\mathbf{R}_\alpha)$ for low rank [3]. In general, the procedure expresses the Poincaré polynomial of $\mathbf{R}_\alpha$ as the difference of two infinite series, the first being determined by the rank, genus, and conjugacy class of $\alpha$ (cf. formulas (13-15) of [3]), and the second being a power series whose coefficients are polynomial functions in $(1+t)^{2g}$. As in the non-parabolic case, this formula results from an equivariantly perfect stratification on the space of holomorphic structures on a fixed topological parabolic bundle over $(F, J)$.

This stratification leads to an algorithm for calculating $M_\alpha(t)$. As mentioned before, such a formula exists on the level of cohomology, and the crucial point is to see that the map induced by $\varphi$ preserves the stratification. This is only true after taking into account the effect of $\varphi$ on the choice of complex structure $J$ on $F$.

Having established such a formula, it is relatively easy to see which unstable strata are created and destroyed as the monodromy condition is allowed to vary. (This is another manifestation of equation $(*)$.) Consequently, equation $(**)$ is seen to follow and this implies part (2).

To prove part (3), consider the case $\omega \in \mathbf{Z}_n$. Define $\widehat{\mathbf{R}}_\omega$ to be the quotient of $\widetilde{\mathbf{R}}_\omega$ by conjugation by the maximal torus of $\mathbf{SU}_n$. It is elementary to see that $\widehat{\mathbf{R}}_\omega$ is diffeomorphic to $\mathbf{R}_\alpha$ in this case, and the proof of part (1) shows that $\mu_\alpha$ is equal to the Lefschetz number of $\varphi^*$ on $\widehat{\mathbf{R}}_\omega$. The natural projection $\psi : \widehat{\mathbf{R}}_\omega \longrightarrow \mathbf{R}_\omega$ is a fiber bundle with fiber $\mathcal{F}$, the flag variety of full flags in $\mathbb{C}^n$, and part (3) now follows by observing that the following diagram commutes

$$\begin{array}{ccc}
\widehat{\mathbf{R}}_\omega & \xrightarrow{\varphi^*} & \widehat{\mathbf{R}}_\omega \\
{\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\
\mathbf{R}_\omega & \xrightarrow{\varphi^*} & \mathbf{R}_\omega .
\end{array}$$

An alternative approach to proving part (3) is to use the identity

$$\mathrm{P}_t(\mathbf{R}_\alpha) = \mathrm{P}_t(\mathcal{F})\mathrm{P}_t(\mathbf{R}_\omega)$$

together with an argument similar to that given for part (2).

*Remarks.* It seems likely that one could define invariants $\mu_\alpha$ for arbitrary (non-fibred) knots using either the approach of [8] or that of [9]. The results of [5] are promising for the computation of the latter invariants. A direct consequence of Theorem 1 is that one of the maps $\phi_\alpha, \phi_\beta$ is a small resolution, hence the intersection homology of $\mathbf{R}_\gamma$ is given by the homology of either $\mathbf{R}_\alpha$ or $\mathbf{R}_\beta$. More generally, if $\gamma$ lies on more than one hyperplane, then $\mathbf{R}_\gamma$ may have a very complicated singular locus $\Sigma_\gamma$, but Conjecture 4.8 of [5] predicts the existence of a small resolution from a nearby non-singular $\mathbf{R}_\alpha$. This conjecture is true if, for example, the hyperplanes containing $\gamma$ are in general position.

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