RATIONALITY OF THE MODULI SPACE OF VECTOR BUNDLES OVER A SMOOTH CURVE

HANS U. BODEN

This article contains a survey on the conjectured rationality of the moduli space of vector bundles over a smooth curve. The main result is a new proof of stable rationality, which implies Conjecture 1 (stated below) for a large number of cases. We describe the progress which had been made on this problem by Tyurin and Newstead, and explain why the proof does not work in general. Ballico rejuvenated interest in the argument, he was the first to prove stable rationality and to recognize its importance. These past results provide the proper historical framework for our contribution to the problem. Because one can read about the details from the original sources [17, 14, 2, 5], the emphasis here is on the central ideas.

After introducing the moduli spaces and stating the conjecture, we digress briefly to discuss the various relevant notions of rationality before presenting the new results. To start, fix:

- X a smooth complex curve of genus $g \geq 1$,
- L a line bundle of degree d, and
- E a holomorphic bundle over X of rank r with $\det E = L$.

The *slope* of E is defined by $\mu(E) = d/r$. E is called *stable* if, for all proper, holomorphic subbundles E' of E, we have $\mu(E') < \mu(E)$, and *semistable* if $\mu(E') \leq \mu(E)$. Given a semistable bundle E, there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

whose quotients $D_i = E_i/E_{i-1}$ are stable with slopes $\mu(D_i) = \mu(E)$. The graded object associated to E is $\operatorname{gr} E = \oplus_{i=1}^s D_i$. Setting $E \sim_S E'$ if $\operatorname{gr} E \cong \operatorname{gr} E'$ defines an equivalence relation on semistable bundles and the moduli space of semistable bundles of rank r with determinant L is by definition the quotient

$$\mathcal{M}_{r,L} = \{E \text{ is semistable, } \operatorname{rank} E = r, \text{ and } \det E = L\}/ \sim_S .$$

The results of [13] show that $\mathcal{M}_{r,L}$ is a normal, projective variety of dimension $(r^2-1)(g-1)$ and that $\mathcal{M}_{r,L}$ is smooth if $(r,d) = 1$.

Conjecture 1. *If r and d are coprime, then $\mathcal{M}_{r,L}$ is rational.*

Suppose that L' is a line bundle of degree d' and consider the map $E \mapsto E \otimes L'$. This obviously preserves semistability and defines an isomorphism $\mathcal{M}_{r,L} \cong \mathcal{M}_{r,L_1}$, where $L_1 = L \otimes (L')^{\otimes r}$. Consequently, the only relevant invariant is the residue class of d modulo r, so we might as well assume $0 < d < r$. Secondly, the map $E \mapsto E^\vee$ defines an isomorphism $\mathcal{M}_{r,L} \cong \mathcal{M}_{r,L^{-1}}$. With regard to the first convention, this interchanges d and $r - d$.

Now recall the two equivalent definitions of rationality, as well as the kindred of notions unirationality and stable rationality.

An n-dimensional variety V over the field k is called *rational* if it is birationally equivalent to projective space, i.e. if there exist rational maps $\phi : V - - \to \mathbb{P}^n$ and $\psi : \mathbb{P}^n - - \to V$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity mappings wherever they are defined. Equivalently, V is rational if and only if the field $k(V)$ of meromorphic functions on V coincides with the purely transcendental field extension $k(x_1, \dots , x_n)$ of k, also known as the field of rational functions.

One way to show a variety is not rational is to exhibit a non-zero birational invariant. It seems unlikely that this can establish the converse, the vanishing of invariants typically only allows one to conclude V is *unirational*. Recall that V is defined to be *unirational* if there exists a dominant rational map $\psi : \mathbb{P}^n - - \to V$. This is equivalent to embedding $k(V)$ in a purely transcendental extension $k(x_1, \dots , x_m)$ of k. The question whether unirational implies rational is an old and famous problem, which for $\dim = 1$ is Lüroth's theorem and for $\dim = 2$ is due to Castelnuovo and Enriques. For $\dim > 2$, it is known to be false, even for $k = \mathbb{C}$ [6, 7].

Now we come to a useful intermediate property introduced by Kóllar and Schreyer, that of stable rationality [8]. V is *stably rational* if $V \times \mathbb{P}^k$ is rational. This is equivalent to the condition that the purely transcendental extension $k(V)(x_1, \dots , x_k)$ of $k(V)$ is a purely transcendental extension of k. The *level* of stable rationality is the smallest k such that $V \times \mathbb{P}^k$ is rational.

Zariski's conjecture asks whether stably rational implies rational. This is also not true for $\dim > 2$, there are examples of stably rational but irrational varieties [4]. The level of stable rationality gives a measure for how far a stably rational variety is from being rational. As we shall see, bounding the level of stable rationality in moduli problems has strong consequences.

Conjecture 1 has an interesting history. In [17], Tyurin proved it in the case $r = 2$ and d odd, and later claimed to prove it in full generality [18]. However, numerous errors were found in [18]. The argument was corrected to some extent by Newstead. His thesis (1966, unpublished) treated the case $r > 2, d = 1 \mod (r)$, and a later article [14] gave a beautiful argument assuming only $(r,d) = 1$. Unfortunately, a small, apparently trivial gap pointed out by Ramanan turned out to be insurmountable and Newstead later retracted his more general statement. Nevertheless, his argument still proved rationality in a number of cases, including:

(i) $d = \pm 1 \mod (r)$,
(ii) $(r,d) = 1$ and $g = p^k$ is a prime power,
(iii) $(r,d) = 1$, p_1 and p_2 are the two smallest distinct prime factors of g and $p_1 + p_2 > r$.

The argument works in many other instances, but determining the set of all triples (g,r,d) for which it holds is complicated and unenlightening.

The culprit in all of this is the universal bundle, or rather its nonexistence. Recall that a universal bundle $\mathcal{U}_{r,L}$ parametrized by $\mathcal{M}_{r,L}$ is a rank r bundle over $\mathcal{M}_{r,L} \times X$ whose restriction to any slice $\{e\} \times X$ is isomorphic to E where $[E] = e$. Of course, such a bundle, if it exists, is only unique up to tensoring by the pullback of a line bundle on $\mathcal{M}_{r,L}$. It is known that such a bundle exists if and only if $(r,d) = 1$ [10, 19, 16].

We now sketch Newstead's argument. Suppose that $0 < d < r$ and $(r,d) = 1$. Set $d_1 = d + r(g-1)$ and suppose $\det E = L_1$, a line bundle of degree d_1. By Serre duality, we see that $h^1(E) = 0$, hence $h^0(E) = d$ by Riemann-Roch. A basis $s_1, \ldots, s_d$ for $H^0(E)$ is likely to be everywhere linearly independent, at least for generic E. In this case, we get the short exact sequence

$$0 \longrightarrow I^d \longrightarrow E \longrightarrow E' \longrightarrow 0, \tag{1}$$

where I^d denotes the trivial bundle of rank d. One would expect further that E' is stable whenever E is stable.

Turning this around, Newstead proved that one can describe "most" stable bundles E as extensions of the form (1) above, where E' ranges over the isomorphism classes of stable bundles of rank $r' = r - d$ and with $\det E' = L_1$. For a fixed stable E', isomorphism classes of such extensions are classified by the quotient of the nonzero elements of

$$H^1(E'^{\vee} \otimes I^d) = H^1(E'^{\vee})^{\oplus d}$$

by the natural action of the automorphism group $\operatorname{Aut} I^d = GL(d)$. (The automorphism group of E' is irrelevant because stable bundles are *simple*.) This quotient space is a Grassmanian, hence rational, and has dimension $\chi = d(g-1)(2r-d)$.

A dimension count shows

$$\begin{aligned}\dim \mathcal{M}_{r,L_1} - \chi &= (r^2-1)(g-1) - d(g-1)(2r-d) \\ &= ((r-d)^2-1)(g-1) = \dim \mathcal{M}_{r-d,L_1},\end{aligned}$$

which is rather convincing that Newstead's idea is indeed correct. In order to make this rigorous, Newstead realized that one must inductively construct a family of bundles using such extensions. In order for the family to have the correct properties, one ultimately needs a family parametrized by $\mathcal{M}_{r-d,L_1}$, or at least by a Zariski-open subset of $\mathcal{M}_{r-d,L_1}$. For non-coprime rank and degree, there is a non-zero obstruction [16], namely the universal bundle does not exist (even over any Zariski-open subset!). Although there exists a projective bundle over the stable points, it is not locally trivial in the Zariski topology. Given a fibration with rational base and rational fiber, in order to conclude the total space is rational, one needs local triviality of the fibration in the Zariski topology. This explains why the condition $(r-d, d_1)$ pops up in the following theorem, which was proved in [14] (though it is not stated precisely this way, cf. [2]).

Theorem 2 (Newstead)**.** *Suppose that* $0 < d < r$ *and* $(r,d) = 1$. *Set* $d_1 = d + r(g-1)$ *and* $\chi = d(g-1)(2r-d)$, *and choose* L_1 *a line bundle of degree* d_1. *If* $(r-d, d_1) = 1$, *then* $\mathcal{M}_{r,L_1}$ *is birationally equivalent to* $\mathcal{M}_{r-d,L_1} \times \mathbb{P}^\chi$.

Newstead's proposal for proving Conjecture 1 is to use induction, the idea being that once $\mathcal{M}_{r',L_1}$ is known to be rational for $r' < r$, rationality of $\mathcal{M}_{r,L_1}$ follows immediately from Theorem 2. The problem is with the inductive hypothesis, namely $(r,d) = 1$, and is illustrated by the following example.

Example 3. *If* $g = 6, r = 5$, *and* $d = 2$, *then* $d_1 = 27, r' = 3$ *and although* $(r,d) = 1$, $(r', d_1) \neq 1$ *and Theorem 2 does not hold. It remains an open question whether* $\mathcal{M}_{5,L}$ *is rational in this case.*

Ballico noticed that Newstead's argument did prove stable rationality of the moduli spaces $\mathcal{M}_{r,L}$ for coprime rank and degree [2]. He also pointed out that stable rationality of $\mathcal{M}_{r-d,L}$ with an appropriate bound on the *level* is enough to conclude rationality of $\mathcal{M}_{r,L}$ from Theorem 2. His argument for stable rationality worked upstairs on the Quot scheme, and for this reason he did not make any systematic progress toward proving Conjecture 1 because the dimension jump was too large to successfully bound the level. (He had an alternative approach, the "up and down" argument at the end of his paper, and this proved rationality in some cases. But as in Newstead's method, determining the triples (g, r, d) for which the argument works leads to intractable numerological problems.)

Our contribution is the following theorem.

Theorem 4. *If $(r, d) = 1$, then $\mathcal{M}_{r,L}$ is stably rational with level $k \leq r - 1$.*

Corollary 5. *Suppose $(r, d) = 1$ and $0 < d < r$. Set $d_1 = d + r(g-1)$ and choose L_1 a line bundle of degree d_1. If either $(r - d, g) = 1$ or $(d, g) = 1$, then $\mathcal{M}_{r,L}$ is rational.*

Remark. This proves Conjecture 1 when $(r, d) = 1$ for all but a thin set of genera g. For example, consider the case $r = 110$ and $d = 43$. Then the corollary implies that $\mathcal{M}_{r,L}$ is rational unless g is a multiple of 2881.

Note that $(r - d, g) = 1$ implies $(r - d, d_1) = 1$, which is what we need in order to be able to apply Theorem 2. The other condition, $(d, g) = 1$, results from an application of the symmetry $E \longleftrightarrow E^\vee$.

Theorem 4 is proved with the help of parabolic bundles. General definitions can be found in [5]. Choose $p \in X$, a complex line F_2 in $F_1 = E_p$, and weights α_1, α_2 with $0 \leq \alpha_1 < \alpha_2 < 1$. This defines a parabolic structure on E making it into a parabolic bundle, which we denote by E_*. The multiplicities of E_* are the numbers $m_1 = \dim F_1/F_2 = r - 1$ and $m_2 = \dim F_2 = 1$, and the slope of E_* is defined by $\mu(E_*) = (d + m_1\alpha_1 + m_2\alpha_2)/r$. The notion of parabolic stability gives rise to the moduli space $\mathcal{M}_{r,L,\alpha}$ of rank r parabolic semistable bundles with multiplicities m_1, m_2, weights $\alpha = (\alpha_1, \alpha_2)$, and determinant L. It follows from [9] that $\mathcal{M}_{r,L,\alpha}$ is a normal, projective variety of dimension $(r^2 - 1)(g - 1) + r - 1$ and that for generic choice of α, $\mathcal{M}_{r,L,\alpha}$ is smooth.

If $(r, d) = 1$, then for any bundle E of rank r and degree d, E semistable implies E is stable. Indeed, for any proper subbundle E' of E, $\mu(E') \neq \mu(E)$. Hence the set $\{\mu(E') \mid E' \subset E\}$ is a discrete subset of $\mathbb{R}$, and by choosing α_1, α_2 small enough, we see that the parabolic bundle E_* is stable if and only its underlying bundle E is stable. This defines a fibration $\mathcal{M}_{r,L,\alpha} \to \mathcal{M}_{r,L}$ whose fiber can be identified with the projective space $\mathbb{P}^{r-1}$. The crucial point is that this fibration is locally trivial in the Zariski topology. To see this, consider the restriction of the universal bundle $\mathcal{U}_{r,L}$ to $\mathcal{M}_{r,L} \times \{p\}$. The associated projective bundle parametrizes a family of parabolic bundles with (small) weights α_1, α_2. Universality of the moduli space produces a morphism identifying this projective bundle with $\mathcal{M}_{r,L,\alpha}$. Local triviality of the bundle implies $\mathcal{M}_{r,L,\alpha}$ is birational to $\mathcal{M}_{r,L} \times \mathbb{P}^{r-1}$, and Theorem 4 is a consequence of the following theorem.

Theorem 6. *$\mathcal{M}_{r,L,\alpha}$ is rational for all α.*

Remark. In [5], rationality of $\mathcal{M}_{r,L,\alpha}$ is proved in many other cases as well, including:

(i) a multiplicity m_i equals one for some i,
(ii) the rank and degree are coprime,
(iii) the rank and degree of any parabolic bundle obtained from E_* by shifting (Definition 5.1, [5]) are coprime.

The strategy for proving the above theorem is similar to Newstead's. The first step is to show that the weights play no role, i.e. that the birational type of $\mathcal{M}_{r,L,\alpha}$ is independent of α. The next step is to prove existence of the universal bundle $\mathcal{U}_{r,L,\alpha}$ parametrized by $\mathcal{M}_{r,L,\alpha}$. This is necessary to avoid the difficulties encountered in the non-parabolic situation. Proposition 3.2 in [5] establishes the existence of $\mathcal{U}_{r,L,\alpha}$ whenever the quasi-parabolic structure admits generic weights, i.e. whenever semistable $\Rightarrow$ stable for some compatible choice of weights.

The assumption $(r, d) = 1$ is replaced by an assumption on the quasi-parabolic structure, namely (i) above, that one of the multiplicities equals one, and the proof boils down to the simpler case presented here.

Assuming $0 \leq d < r$, the argument splits into two cases, $d > 0$ or $d = 0$. In the first case, one can show just as before that "most" stable parabolic bundles E_* are described by extensions of the form

$$0 \longrightarrow I_*^d \longrightarrow E_* \longrightarrow E'_* \longrightarrow 0, \tag{2}$$

where I_*^d is the trivial parabolic bundle of rank d with weight α_1 and E'_* is allowed to vary among isomorphism classes of parabolic stable bundle of rank $r-d$ with multiplicities $m'_1 = r-d-1$ and $m'_2 = 1$. This case then follows by induction once the following theorem is established.

Theorem 7. *Suppose that $0 < d < r$ and $\alpha = (\alpha_1, \alpha_2)$ is generic. Set $d_1 = d + r(g-1)$ and $\chi = d(g-1)(2r-d) + d$ and choose L_1 a line bundle of degree d_1. Then $\mathcal{M}_{r,L_1,\alpha}$ is birationally equivalent to $\mathcal{M}_{r-d,L_1,\alpha} \times \mathbb{P}^\chi$.*

For the remaining case, $d = 0$, we use the Hecke correspondence for parabolic bundles. This correspondence was introduced as a method to use parabolic bundles to pass between two moduli spaces of non-parabolic bundles of different degree [12, 9]. As such, it includes going up and down projective fibrations which are not necessarily locally trivial in the Zariski topology (as usual, this problem arises in the case of non-coprime rank and degree). Hence the Hecke correspondence is not useful in establishing the rationality conjecture. For parabolic bundles, the Hecke correspondence does not involve any such fibrations, so it is a much more valuable tool in proving rationality of $\mathcal{M}_{r,L,\alpha}$. The quasi-parabolic structure allows for direct comparison of moduli spaces of parabolic bundles of different degree, in some sense it allows one to control the amount of energy bubbled off. In the example under consideration, given E_*, the Hecke correspondence produces a parabolic bundle E'_* of rank r, weights $\alpha' = (\alpha'_1, \alpha'_2)$, multiplicities $m_1 = 1$ and $m_2 = r-1$, and determinant L' of degree $d' = d-r+1$. This determines an isomorphism $\mathcal{M}_{r,L,\alpha} \cong \mathcal{M}_{r,L',\alpha'}$. We have already seen that $\mathcal{M}_{r,L',\alpha'}$ is birational to $\mathcal{M}_{r,L'} \times \mathbb{P}^{r-1}$, but $d = 0 \Rightarrow d' \equiv -1 \mod (r)$ so case (i) of Newstead's result shows $\mathcal{M}_{r,L'}$, and hence $\mathcal{M}_{r,L',\alpha'}$, is rational.

In conclusion, we mention a few open problems. If $(r,d) \neq 1$, the problem of rationality of $\mathcal{M}_{r,L}$ is also interesting. Beyond the evident unirationality, however, there is almost no support for Conjecture 1 in this case. For example, stable rationality of these moduli spaces is at the time of writing an open problem. The difficulties are non-smoothness of the moduli space and non-existence of the universal bundle. The place to start is rank two and degree zero because for low genus there are rather explicit descriptions of the moduli space. In fact, for $g = 2, \mathcal{M}_{r,L} = \mathbb{P}^3$ [12].

On the other hand, the overall evidence for the rationality conjecture in the coprime case is now quite overwhelming. The remaining cases will probably succumb to demonstration and Newstead's beautiful argument will then be fully restored. In contemplating this, one is lead to ponder whether a restricted version of the Zariski conjecture also fails for dimension $n > 2$. This stems largely from the success achieved by bounding the level of stable rationality. On the one hand, from the examples of [4], it follows that there exist irrational varieties V with $V \times \mathbb{P}^1$ rational (cf. problem 6, [3]). But if one restricts attention to, say, varieties of dimension n, or perhaps an even more exclusive class, can one establish Zariski's conjecture assuming an appropriate bound on the level of stable rationality?

References

[1] AMS/IP Studies in Advanced Mathematics, ed. by W. Kazez, Amer. Math. Soc. and Internat. Press, 1997.

[2] E. Ballico, *Stable rationality for the variety of vector bundles over an algebraic curve,* J. Lond. Math. Soc. (2), **30**, (1984), 21 –26.

[3] A. Beauville, *Variétés rationnelles et unirationelles,* in Algebraic Geometry-Open Problems, LNM **997**, Springer-Verlag, 1983.

[4] A. Beauville, J.-L. Colliot-Thèléne, J.-J. Sansuc and P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles,* Ann. of Math. **121** (1985), 283–318.
[5] H. U. Boden and K. Yokogawa, *Rationality of moduli spaces of parabolic bundles,* to appear in J. Lond. Math. Soc. (1997), `alg-geom 9610013`.
[6] H. Clemens and P. Griffiths, *The intermediate Jacobian of the cubic threefold,* Ann. of Math. **95** (1972), 281–356.
[7] V. Iskovkih and Yu. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem,* Math. USSR-Sbornik, **15** (1971), 141–166.
[8] J. Kollár and F.-O. Schreyer, *The moduli space of curves is stably rational for $g \leq 6$,* Duke Math. J., **51**, (1984), 239–242.
[9] V. Mehta and C. Seshadri, *Moduli of vector bundles on curves with parabolic structures,* Math. Ann., **248**, (1980), 205–239.
[10] D. Mumford and P. Newstead, *Periods of a moduli space of bundles on curves,* Amer. J. Math., **90**, (1968), 1200–1208.
[11] M. Narasimhan and S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math., **89**, (1969), 14–51.
[12] M. S. Narasimhan and S. Ramanan, *Geometry of Hecke Cycles – I,* C. P. Ramanujan, A Tribute, T.I.F.R. Bombay, 1978.
[13] M. Narasimhan and C. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math., **82**, (1965), 540–564.
[14] P. E. Newstead, *Rationality of moduli spaces of stable bundles,* Math. Ann., **215**, (1975) 251–268; *Correction,* ibid., **249**, (1980) 281–282.
[15] P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces,* Tata Institute, Bombay, 1978.
[16] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve,* Math. Ann., **200**, (1973) 69–84.
[17] A. N. Tyurin, *On the classification of two-dimensional vector bundles over an algebraic curve of arbitrary genus,* Izv. Akad. Nauk. SSSR Ser. Mat. **28**, (1964), 21 – 52.
[18] A. N. Tyurin, *Classification of vector bundles over an algebraic curve of arbitrary genus,* Izv. Akad. Nauk. SSSR Ser. Mat. **29**, (1965), 657 – 688. English translation, Amer. Math. Soc. Transl. **63**, (1967), 245 – 279.
[19] A. N. Tyurin, *Analogues of Torelli's theorem for 2-dimensional bundles over algebraic curves of arbitrary genus,* Math. USSR-Izv., **3** (1969), 1081–1101.

Ohio State University-Mansfield, Mansfield, OH
E-mail address: `boden@math.ohio-state.edu`