INTEGRALITY OF THE AVERAGED JONES POLYNOMIAL OF ALGEBRAICALLY SPLIT LINKS

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This note arose out of an attempt to prove a conjecture of Lin and Wang concerning the the integrality of the coefficients of the Taylor series expansion at $t = 1$ of the averaged Jones polynomial of algebraically split links. This question comes up in the study of Ohtsuki's invariants, $\lambda_n(M) \in \mathbb{Q}$, defined for M an integral homology 3-sphere [O, LW].

Suppose that L is an oriented link in S^3 with μ components. Write the averaged Jones polynomial of L as a Taylor series at $t = 1$, i.e.

$$\Phi(L;t) = \sum_{i=0}^{\infty} a_i(t-1)^i,$$

and set $\phi_n(L) = (-2)^\mu a_{n+\mu}(L)$.

Conjecture 4.1 of [LW] states that for L an algebraically split link (ASL),

$$n!\,\phi_n(L) \in 6\mathbb{Z}.$$

This conjecture is verified for $n = 1, 2$ in [LW], and we consider the case $n \geq 3$ here. We first establish that $a_n(L) \in \mathbb{Z}$ whenever L is a geometrically split link (GSL), implying that $\phi_n(L) \in 2^\mu\mathbb{Z}$, which is a priori stronger than the conjecture in this case. Nevertheless, Conjecture 4.1 is not true for ASLs. The problem is the presence of additional factors of 2 in the denominator of $\phi_n(L)$.

The goal of this paper is to present two results, Proposition 1 for GSLs and Proposition 2 for ASL, both giving integrality results for the coefficients $a_n(L)$ which are sharp, as shown by examples (I) and (II).

For the definition of $\Phi(L;t)$, see pp.10–11 of [LW]. It is roughly a sum of (normalized) Jones polynomials, summed over *all* sublinks of L, and it satisfies:

(i) If L is trivial or empty, then $\Phi(L;t)=1$.

(ii) If $L=L_1\cup L_2$ is a geometric splitting, then $\Phi(L_1\cup L_2;t)=\Phi(L_1;t)\cdot\Phi(L_2;t)$.

This Jones polynomial satisfies a skein relation slightly different from the usual one and is normalized by dividing by $(t^{1/2}+t^{-1/2})^\mu$ (see p.11, [LW]). For example, if L is Brunnian (i.e. all proper sublinks are trivial) and $J(L;t)$ is the usual Jones polynomial (i.e. the one tabulated in C. Adams' book [A]), then

$$\Phi(L;t)=(-1)^\mu\left(1-\frac{J(L;1/t)}{(t^{1/2}+t^{-1/2})^\mu}\right).$$

This and property (ii) is all we need to know to settle the conjecture.

Define

$$a_n(L)=\frac{1}{n!}\left.\frac{d^n\Phi(L;t)}{dt^n}\right|_{t=1}.$$

We claim that $a_n(L)\in\mathbb{Z}$ for L a GSL. For knots K, it is evident that $\Phi(K;t)$ is a Laurent polynomial, and property (ii) implies the same for L a GSL, i.e. $\Phi(L;t)\in\mathbb{Z}[1/t,t]$. Obviously the nth derivative of t^m at $t=1$ is simply $m(m-1)\cdots(m-n+1)$, which is divisible by $n!$, since any product of n successive integers is divisible by $n!$ (p. 63 [HW]). It now follows by linearity that $\left.\frac{d^n\Phi(L;t)}{dt^n}\right|_{t=1}$ is divisible by $n!$, hence $a_n(L)\in\mathbb{Z}$. This implies Conjecture 4.1 for GSLs, but in fact, more is true.

For any ASL L, Theorem 4.1 and Lemma 4.2 of [LW] show that

$$\Phi(L;t)=\sum_{i=\mu+1}^{\infty}a_i(K)(t-1)^i, \tag{1}$$

(i.e. $a_i(L)=0$ for $i\le\mu$), and that both $a_{\mu+1}(L)$ and $2a_{\mu+2}(L)\in 3\mathbb{Z}$. But if $\mu=1$ and $L=K$ is a knot, then $a_n(K)\in\mathbb{Z}$, hence $a_3(K)\in 3\mathbb{Z}$.

Proposition 1. *Suppose $L=K_1\cup\cdots\cup K_\mu$ is a GSL. Then*

$$\Phi(L;t)=\sum_{i=2\mu}^{\infty}a_i(L)(t-1)^i,$$

where

$$\begin{aligned}
&a_i(L)\in 3^\mu\mathbb{Z} && \text{for } 2\mu\le i\le 3\mu, \text{ and}\\
&a_i(L)\in 3^{4\mu-i}\mathbb{Z} && \text{for } 3\mu< i\le 4\mu,\\
&a_i(L)\in \mathbb{Z} && \text{for } 4\mu< i.
\end{aligned}$$

Proof. By property (ii), we have

$$\Phi(L;t) = \Phi(K_1;t)\cdots\Phi(K_\mu;t).$$

Multiplication of the series expansions of $\Phi(K_j;t) = \sum_{i=0}^{\infty} a_i(K_j)(t-1)^i$ gives the formula

$$a_i(L) = \sum_{\sigma_1+\cdots+\sigma_\mu=i} a_{\sigma_1}(K_1)\cdots a_{\sigma_\mu}(K_\mu).$$

The proposition now follows from the fact that $a_0(K_j) = 0 = a_1(K_j)$, and that $a_2(K_j)$ and $a_3(K_j)$ are multiples of 3 for $j = 1, \ldots, \mu$. $\square$

Examples. (I) If K is the left-hand trefoil, then

$$\begin{aligned} \Phi(K;t) &= -t^4 + t^3 + t - 1 \\ &= -3(t-1)^2 - 3(t-1)^3 - 1(t-1)^4. \end{aligned}$$

Taking L to be the GSL $K \cup \cdots \cup K$ shows that Proposition 1 is sharp.

(II) Let L be the Whitehead link. Consulting link tables[1], we obtain

$$\begin{aligned} \Phi(L;t) &= \frac{-t^{7/2} + 2t^{5/2} - t^{3/2} + 2t^{1/2} - t^{-1/2} + t^{-3/2}}{t^{1/2} + t^{-1/2}} - 1 \\ &= -t^3 + 3t^2 - 4t + 5 + t^{-1} - 8(t+1)^{-1}. \\ &= \frac{-3(t-1)^3}{2} + \sum_{n=4}^{\infty} \frac{(-1)^n(2^{n-2}-1)(t-1)^n}{2^{n-2}}. \end{aligned}$$

In particular, $a_5(L) = -7/8$ and so $3!\,\phi_3(L) = 21$, which provides a counterexample to Conjecture 4.1. Notice moreover that $\phi_7(L) = \frac{127}{32}$, thus $n!\,\phi_n(L)$ need not be an integer.

Proposition 2. *If L is an ASL with μ components, then*

$$2^{n-2} a_n(L) \in \mathbb{Z}.$$

[1] $L = 5_1^2$ in the standard notation [A]. Note that $J(L;t)$ depends on a choice of orientation.

Proof. For L a GSL, this is a consequence of Proposition 1, while for L an ASL, it follows by induction on n, μ, and the double unlinking number, which is the number of double crossings needed to change L to a GSL, as we now explain.

Consider two crossings, one positive and the other negative, between distinct components of L. Write $L = L_{+-}$, and notice that L_{-+} is an ASL with μ components, and that L_{0+} and L_{+0} are also ASL links, but with $\mu - 1$ components. Using the skein relation twice and subtracting, we obtain the double crossing change formula (cf. Lemma 4.1, [LW])

$$(t+1)\big(\Phi(L_{+-};t) - \Phi(L_{-+};t)\big) = (t^2 - t)\big(\Phi(L_{0+};t) - \Phi(L_{+0};t)\big).$$

Equating coefficients of the power series in equation (1) gives

$$\begin{aligned} a_n(L_{+-}) \quad &= a_n(L_{-+}) + \frac{a_{n-1}(L_{-+}) - a_{n-1}(L_{+-})}{2} \\ + \qquad & \frac{a_{n-1}(L_{0+}) - a_{n-1}(L_{+0})}{2} + \frac{a_{n-2}(L_{0+}) - a_{n-2}(L_{+0})}{2}. \end{aligned}$$

The proposition now follows from this formula by induction, since we can assume that it has already been established for L_{-+}, L_{0+}, and L_{+0}, and that $2^{n-3}a_{n-1}(L_{+-}) \in \mathbb{Z}$. Note that the previous example indicates that this result is sharp. $\square$

References

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