RATIONALITY OF MODULI SPACES OF PARABOLIC BUNDLES

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Abstract. The moduli space of parabolic bundles with fixed determinant over a smooth curve of genus greater than one is proved to be rational whenever one of the multiplicities associated to the quasi-parabolic structure is equal to one. It follows that if rank and degree are coprime, the moduli space of vector bundles is stably rational, and the bound obtained on the level is strong enough to conclude rationality in many cases.

1. Introduction

Let X be a smooth complex curve of genus $g \geq 2$, L a line bundle of degree d over X, and $\mathcal{M}_{r,L}$ the moduli space of semistable bundles E of rank r with determinant L.

Conjecture 1.1. *$\mathcal{M}_{r,L}$ is rational, i.e. it is birational to a projective space.*

Despite many positive results [12], this is still an open problem, even for $(r,d) = 1$.

In this paper, we study a closely related problem, namely the birational classification of moduli spaces of parabolic bundles over X. These moduli spaces occur naturally as

(i) unitary representation spaces of Fuchsian groups [10],
(ii) moduli spaces of Yang-Mills connections on X with an orbifold metric [5], and
(iii) moduli spaces of certain semistable bundles over an elliptic surface [3].

The theory developed in [7] and extended here shows that their birational type depends only on the *quasi-parabolic* structure (see Proposition 4.3). The methods of [12] then prove, in many cases, that these moduli spaces are rational. The weaker result, Theorem 6.1, uses only Newstead's theorem, while the stronger one, Theorem 6.2, requires an adaptation of his inductive argument.

Using the theory developed in §4, it then follows from Theorem 6.2 that $\mathcal{M}_{r,L} \times \mathbb{P}^{r-1}$ is rational if $(r,d) = 1$ (see Corollary 6.4). Stable rationality of the moduli spaces had been proved in this case by Ballico [2], and our result is a strengthening of his. For instance, a consequence is that one can conclude Conjecture 1.1 under the assumption that $(r,d) = 1$ for most values of the genus[1] (see Corollary 6.5).

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[1]Choosing $d' \equiv d \mod (r)$ with $0 < d' < r$, the hypothesis is that either $(d', g) = 1$ or $(r - d', g) = 1$.

A number of useful facts are established along the way. One key point is Proposition 3.2, which gives a simple criterion for the existence of a universal bundle of stable parabolic bundles. We also extend the theory developed in [7] in several important ways (Theorems 4.1, 4.2, and 5.3); the first two are standard but necessary for our purposes and the third is completely new. Its proof requires the idea of shifting a parabolic sheaf (Definition 5.1), which also provides a framework for the Hecke correspondence (equation (10)). All of these results play a crucial role in the proofs of Theorems 6.1 and 6.2.

A brief word about the organization of this paper: §2 introduces the notation used in the following sections, §3 discusses the existence of universal families, §4 summarizes and extends the theory of [7], §5 describes shifting and the Hecke correspondence, and §6 contains the proofs of the main results and their corollaries.

Before we begin, we would like to acknowledge a certain debt to the work of Newstead, upon which a number of our arguments depend, and without which this paper would be inconceivable.

2. Notation

Let X be a smooth curve of genus $g \geq 2$ and D a reduced divisor in X. If E is a $\mathbb{C}^r$ bundle over X, then a *parabolic structure* on E with respect to D is just a collection of weighted flags in the fibers of E over each $p \in D$ of the form

$$E_p = F_1(p) \supset F_2(p) \supset \cdots \supset F_{\kappa_p}(p) \supset 0, \tag{1}$$

$$0 \leq \alpha_1(p) < \alpha_2(p) < \cdots < \alpha_{\kappa_p}(p) < 1. \tag{2}$$

Holomorphic bundles E with parabolic structures are called *parabolic bundles*, and we use the notation E_* to indicate the bundle (or, equivalently, locally-free sheaf) E together with a choice of parabolic structure. A morphism $\phi : E_* \longrightarrow E'_*$ of parabolic bundles is a bundle map satisfying $\phi(F_i(p)) \subset F'_{j+1}(p)$ whenever $\alpha_i(p) > \alpha'_j(p)$ for all $p \in D$. We use the tensor product notation $H^0(E_*^\vee \otimes E'_*)$ for these morphisms, where $E_*^\vee$ denotes the dual parabolic bundle (cf. [18]).

A *quasi-parabolic* structure on E is what is left after the weights are forgotten, it is determined topologically by its flag type m, which specifies *multiplicities* $m(p) = (m_1(p), \ldots, m_{\kappa_p}(p))$ for each $p \in D$ defined by $m_i(p) = \dim F_i(p) - \dim F_{i+1}(p)$.

A subbundle E' inherits a parabolic structure from one on E in a canonical way: The flag in E'_p is gotten by intersecting with the flag in E_p and the weights are determined by choosing maximal weights such that the inclusion map from E' to E is parabolic (p. 213, [10]). Parabolic structures on quotients have a similar description (loc. cit.).

A parabolic bundle E_* is called *stable* if every proper holomorphic subbundle E' satisfies $\mu(E'_*) < \mu(E_*)$, where

$$\mu(E_*) = \operatorname{pardeg} E_*/r = \deg E/r + \sum_{p\in D}\sum_{i=1}^{\kappa_p} m_i(p)\alpha_i(p)/r.$$

The parabolic bundle E_* is called *semistable* if $\mu(E'_*) \leq \mu(E_*)$ for each subbundle E'_*. The construction of the moduli space $\mathcal{M}_\alpha$ of semistable parabolic bundles, as a normal, projective variety, is given in [10]. The subspace $\mathcal{M}^s_\alpha$ of stable bundles is smooth and Zariski-open, in particular, if every semistable bundle is stable, then $\mathcal{M}_\alpha$ is smooth.

Let $\Delta^r = \{(a_1, \dots, a_r) \mid 0 \leq a_1 \leq \cdots \leq a_r < 1\}$ and define $W = \{\alpha : D \longrightarrow \Delta^r\}$. Points in W determine both the weights and the multiplicities. Conversely, given a weight α in the sense of (2), the associated point in W is gotten by repeating each $\alpha_i(p)$ according to its multiplicity $m_i(p)$. We abuse notation slightly by referring to points in W as weights. This gives an obvious notion of when a weight is *compatible* with a choice of multiplicities, and for a given m, we define the open face of weights compatible with m to be

$$V_m = \{\alpha \in W \mid \alpha_{i-1}(p) = \alpha_i(p) \Leftrightarrow \sum_{k=1}^{j} m_k(p) < i \leq \sum_{k=1}^{j+1} m_k(p)\}.$$

A weight in the interior of W specifies full flags at each $p \in D$. For every other choice of m, V_m is contained in the boundary of W. Now W is a simplicial set, and the face relations give a natural ordering on $\{V_m\}$ and we write $V_m > V_{m'}$ if $V_{m'}$ is a proper face contained in the closure of V_m. This agrees with the natural ordering on m gotten by successive refinement.

Weights for which $\mathcal{M}_\alpha$ is not necessarily smooth satisfy $\mu(E'_*) = \mu(E_*)$ for some proper subbundle E'. Letting E'' be the quotient, then the short exact sequence of parabolic bundles $E'_* \xrightarrow{\iota} E_* \xrightarrow{\pi} E''_*$ determines a partition of (d, r, m) in the obvious way: (d', d''), (r', r'') and (m', m'') are the degrees, ranks, and multiplicities of (E', E''). (We define m' and m'' here slightly unconventionally, namely

$$\begin{aligned} m'_i(p) &= \dim(F_i(p) \cap \iota(E'_p)) - \dim(F_{i+1}(p) \cap \iota(E'_p)), \\ m''_i(p) &= \dim(\pi(F_i(p)) \cap E''_p) - \dim(\pi(F_{i+1}(p)) \cap E''_p), \end{aligned}$$

for $p \in D$ and $1 \leq i \leq \kappa_p$.) Notice that $r', r'' > 0$ and $m'_i(p), m''_i(p) \geq 0$. Write $\xi = (d', r', m')$. For fixed ξ, the set of weights compatible with m for which $\mu(E'_*) = \mu(E_*)$ is the intersection of a hyperplane H_ξ in W with V_m given by the equation

$$\sum_{i=1}^{\kappa_p} m_i(p)\alpha_i(p)) \sum_{p\in D}\sum_{i=1}^{\kappa_p}(r' m_i(p) - r m'_i(p))\alpha_i(p) = rd' - r'd. \tag{3}$$

There are only finitely many hyperplanes; the above equation puts a bound on d' and all other quantities are already bounded. We shall refer to $H_\xi \cap V_m$ as a *wall* in V_m. These walls induce a chamber structure on V_m, a *chamber* being a connected component of $V_m \setminus \cup_\xi H_\xi$ (it is possible that $V_m \subset H_\xi$). Weights $\alpha \in W \setminus \cup H_\xi$ are called *generic*, and for these weights, $\mathcal{M}_\alpha = \mathcal{M}_\alpha^s$. In the next section, we shall see that V_m contains a generic weight if and only if the degree d and the set of multiplicities $\{m_i(p)\}$ have greatest common divisor equal to one.

3. Families of parabolic bundles

In this section, we present Proposition 3.2, which establishes the existence of a universal family of stable parabolic bundles parametrized by $\mathcal{M}_\alpha^s$ whenever V_m contains a generic weight. Although results of this type are well-known to experts, the proposition, as well as the proof, are original (cf. Théorème 32, [15]). It is important because, in the case of ordinary bundles, the non-existence of the universal family ([14]) is the obstruction to proving Corollary 1.1 by induction, and as shown in §6, the analogous argument works for parabolic bundles precisely because the necessary conditions for the vanishing of this obstruction given by Proposition 3.2 are often satisfied.

Given positive integers $m_1, \dots, m_\kappa$ such that $m_1 + \cdots m_\kappa = r$, define $\mathcal{F}_m$ to be the variety of flags of type m. These are simply flags $\mathbb{C}^r = F_1 \supset \cdots \supset F_s \supset 0$ with $\dim F_i - \dim F_{i+1} = m_i$. Furthermore, for any bundle $E \longrightarrow S$ of rank r, let $\mathcal{F}_m(E) \longrightarrow S$ be the bundle of flags of type m.

Given a bundle $U \to S \times X$, we adopt the notation $U_s = U|_{\{s\}\times X}$. We also use π_S for the projection map $S \times X \to S$.

Definition 3.1. *Fix multiplicities $m(p)$ for each $p \in D$.*

(i) *A family of quasi-parabolic bundles (of type m) parametrized by a variety S is a bundle U over $S \times X$ together with a section ϕ_p of the flag bundle $\mathcal{F}_{m(p)}(U|_{S\times\{p\}}) \longrightarrow S$ for each $p \in D$.*

(ii) *Two families (U, ϕ) and (U', ϕ') parametrized by S are equivalent, written $(U, \phi) \sim (U', \phi')$, if there exists a line bundle L over S and an isomorphism $U \cong U' \otimes \pi_S^* L$ under which $\phi \mapsto \phi'$.*

Note that the section ϕ_p in (i) above is just a choice of a nested chain of subbundles of $U|_{S\times\{p\}}$ whose relative coranks are given by the multiplicities $m(p)$. A family of parabolic bundles is gotten by associating a fixed set of weights to each chain of subbundles. Let $U_* = (U, \phi, \alpha)$ be the resulting family of parabolic bundles and $U_{s,*} = (U_s, \phi(s), \alpha)$ be the parabolic bundle above $s \in S$. Then U_* is called a family of (semi)stable parabolic bundles if $U_{s,*}$ is (semi)stable for each $s \in S$.

It follows from the construction of Mehta and Seshadri that $\mathcal{M}_\alpha$ is a coarse moduli space. Proposition 1.8 of [13] then gives two conditions which are necessary and sufficient for a coarse moduli space to be fine, i.e. to admit a universal family. The second condition is not difficult to verify using an argument similar to that given in Lemma 5.10 of [13]. The first condition requires that we construct a family $\mathcal{U}_*^\alpha$ parametrized by $\mathcal{M}_\alpha^s$ with the property that $\mathcal{U}_{e,*}^\alpha$ is a parabolic stable bundle isomorphic to E_* for all $[E_*] = e \in \mathcal{M}_\alpha^s$.

To construct this family, we need to review the construction of $\mathcal{M}_\alpha$ ([9], [10]). Let Q be the Hilbert scheme of coherent sheaves over X which are quotients of $\mathcal{O}_X^{\oplus N}$ with fixed Hilbert polynomial (that of $E(k)$ for $k \gg g$), where $N = h^0(E)$. Let U be the universal family on $Q \times X$. Define R to be the subscheme of Q of points $r \in Q$ so that U_r is a locally free sheaf which is generated by its global sections and $h^1(U_r) = 0$. Let $\widetilde{R}$ be the total space of the universal flag bundle over R with flag type $\prod_{p \in D} \mathcal{F}_{m(p)}$, and let $\widetilde{U}$ be the pullback of U to $\widetilde{R}$. Then $\widetilde{U}$ is canonically a family of parabolic bundles parametrized by $\widetilde{R}$ by letting, for each $p \in D$, ϕ_p be the tautological section and $\alpha(p)$ be the fixed weights. It follows that $\widetilde{R}$ has the local universal property for parabolic bundles (p. 16, [9]).

The subsets $\widetilde{R}^s$ ($\widetilde{R}^{ss}$) corresponding to the stable (semistable) parabolic bundles are invariant under the natural action of $\mathrm{GL}(N) = \mathrm{Aut}(\mathcal{O}_X^{\oplus N})$, and $\mathcal{M}_\alpha$ is a good quotient of $\widetilde{R}^{ss}$ (with linearization induced by the weights α), and $\mathcal{M}_\alpha^s$ is the geometric quotient of $\widetilde{R}^s$.

The center of $\mathrm{GL}(N)$ acts trivially on R and $\widetilde{R}$, but nontrivially on the locally universal bundle $\widetilde{U}$. In fact, $\lambda(\mathrm{id})$ acts on $\widetilde{U}$ by scalar multiplication by λ in the fibers (this follows from p. 138, [13]). Given a line bundle L over $\widetilde{R}^s$ with a natural lift of the $\mathrm{GL}(N)$ action such that $\lambda(\mathrm{id})$ acts by multiplication by λ, then using $\widetilde{U}^s$ to denote $\widetilde{U}|_{\widetilde{R}^s \times X}$, the quotient of $\widetilde{U}^s \otimes \pi_{\widetilde{R}^s}^* L^{-1}$, together with the tautological sections and weights $\{\phi_p, \alpha(p) \mid p \in D\}$ mentioned above, gives the desired family.

Proposition 3.2. *Such a line bundle L exists if either*

(i) *the elements of the set $\{d, m_i(p) \mid p \in D, 1 \le i \le \kappa_p\}$ have greatest common divisor equal to one, or*

(ii) *the face V_m containing α contains a generic weight.*

Moreover, these two conditions are equivalent, and when they are satisfied, the moduli space $\mathcal{M}_\alpha^s$ is fine.

The idea of the proof is to find line bundles L_k for each $k \in \{d, m_i(p)\}$ over $\widetilde{R}^s$ with natural actions of $\mathrm{GL}(N)$ such that $\lambda(\mathrm{id})$ acts by scalar multiplication by λ^k. Then (i) gives the existence of $k_1, \ldots, k_\ell \in \{d, m_i(p)\}$ and integers $a_1, \ldots, a_\ell$ so that $a_1k_1 + \cdots a_\ell k_\ell = 1$. The required line bundle is then the tensor product $L = L_{k_1}^{a_1} \otimes \cdots \otimes L_{k_\ell}^{a_\ell}$. At the end of the proof, we will show that (i) and (ii) are equivalent.

We start with a lemma.

Lemma 3.3. *Suppose E_* is parabolic semistable or degree d and rank r and H_* is a parabolic line bundle of degree h, then*

$$h^1(H_*^\vee \otimes E_*) \neq 0 \quad \Rightarrow \quad d \leq r(2g-2+h) + r^2 n. \tag{4}$$

Proof. Serre duality for parabolic bundles (Proposition 3.7 of [18]) implies that

$$h^1(H_*^\vee \otimes E_*) \leq h^0(E_*^\vee \otimes H_* \otimes K(D)).$$

(If we had used $h^0(E_*^\vee \otimes \widehat{H}_* \otimes K(D))$, the circumflex over H_* indicating *strongly* parabolic morphisms, we would get the usual statement of Serre duality with equality, cf. [18, 8].) Suppose that $\phi : E \longrightarrow H \otimes K(D)$ is a non-zero map and let E' be the subbundle generated by $\operatorname{Ker}\phi$. Then

$$\deg E' \geq \deg E - \deg H \otimes K(D) = d - h - (2g-2+n).$$

Considering E'_* with its canonical parabolic structure as a subbundle of rank $r-1$, the inequality (4) follows easily from this, semistability of E_*, and the inequalities $\operatorname{pardeg} E'_* \geq \deg E'$ and $\operatorname{pardeg} E_* \geq \deg E + rn$. $\square$

Proof of Proposition. Write the weights α without repetition. Choose $\ell : D \longrightarrow \mathbb{Z}$ with $1 \leq \ell_p \leq \kappa_p + 1$ and set $\beta(p) = \alpha_{\ell_p}(p)$. (Take $\beta(p) > \alpha_{\kappa_p}$ if $\ell_p = \kappa_p + 1$.) For $h \in \mathbb{Z}$, define

$$\chi(\ell, h) = d + r(1-g-h) - \sum_{p \in D} \sum_{i=1}^{\ell_p - 1} m_i(p).$$

Let H_* be the parabolic line bundle with $\deg H = h < d/r - rn - (2g-2)$ and with weights $\beta(p)$ at $p \in D$. It follows from the lemma that if E_* is semistable, then $h^1(H_*^\vee \otimes E_*) = 0$. Thus $h^0(H_*^\vee \otimes E_*) = \chi(\ell, h)$ by Riemann-Roch. Hence $(R^0\pi_{\widetilde{R}^s})(\widetilde{U}^s \otimes \pi_X^* H_*)$ is a locally free sheaf of rank $\chi(\ell, h)$ over $\widetilde{R}^s$. Let $L(\ell, h)$ be the determinant of the corresponding bundle. By construction, the $\mathrm{GL}(N)$ action on $\widetilde{U}$ induces one on this bundle (and hence on $L(\ell, h)$); $\lambda(\mathrm{id})$ acts by scalar multiplication by λ on the bundle and by $\lambda^{\chi(\ell,h)}$ on $L(\ell, h)$. It is now a simple exercise in high school algebra to see that we can choose h, h' and ℓ, ℓ' so that $\lambda(\mathrm{id})$ acts on $L(\ell, h) \otimes L(\ell', h')$ by λ^k for any $k \in \{d, m_i(p)\}$.

This proves the conclusion of the proposition assuming (i), and now we show that conditions (i) and (ii) are equivalent. Suppose first that (i) does not hold. Consider E_* as a quasi-parabolic bundle without holomorphic structure, which will be specified later. Since the set $\{d, m_i(p)\}$ is not relatively prime, there exists a prime number q evenly dividing each element of the set. Clearly q also divides r. Set $d' = d/q, r' = r/q$ and $m_i'(p) = m_i(p)/q$. Consider the quasi-parabolic bundle E_*' with degree d', rank r', and multiplicities m'. Any choice of weights α on E_* induces (the same!) weights on E_*', and it follows that since $g \geq 2$, there is some holomorphic structure for which E_*' is semistable. Define the holomorphic structure on E_* by

$$E_* = E_*' \oplus \overset{q}{\cdots} \oplus E_*'.$$

It follows that E_* is semistable but not stable for *any* choice of compatible weights. This implies that V_m does not contain a generic weight.

Suppose conversely that V_m does not contain a generic weight. Since V_m is affine, $V_m \subset H_\xi$ for some $\xi = (r', d', m')$ Using (3), we conclude that for all $\alpha \in V_m$,

$$\sum_{p \in D} \sum_{i=1}^{\kappa_p} (r m_i'(p) - r' m_i(p)) \alpha_i(p) = r d' - r' d.$$

(Here, we are still thinking of α without repetition.) We can vary each $\alpha_i(p)$ continuously by some small amount, and it follows that

$$r m_i'(p) - r' m_i(p) = 0 = r d' - r' d$$

for all i and p. Since $r' < r$, there exists a prime q such that q^k divides r but not r'. Hence q divides d and each element of the set $\{m_i(p) \mid p \in D, 1 \leq i \leq \kappa_p\}$. $\square$

4. The variation and degeneration theorems

In this section, we describe and extend the theory of [7]. This allows us to compare the moduli spaces of parabolic bundles $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ when

(i) $\alpha, \beta \in V_m$ are generic weights in adjacent chambers,

(ii) $\alpha \in V_\ell$ and $\beta \in V_m$ are generic weights not separated by any hyperplanes and $V_\ell > V_m$.

Cases (i) and (ii) correspond to Theorem 3.1 and Proposition 3.4 of [7]. We present slightly stronger versions of those results tailored for our purposes here.

Starting with (i), suppose that $\alpha, \beta \in V_m$ are generic weights separated by a single hyperplane H_ξ. Choose $\gamma \in H_\xi$ on the straight line connecting α to β. Then $\mathcal{M}_\gamma$ is stratified by the Jordan-Hölder type of the underlying bundle, and since γ lies on only one hyperplane, there are exactly two strata: the stable bundles $\mathcal{M}_\gamma^s$ and the strictly semistable bundles Σ_γ. Writing $\xi = (r', d', m')$ for the partition, then it is not hard to see that $\Sigma_\gamma \cong \mathcal{M}_{\gamma'} \times \mathcal{M}_{\gamma''}$, with the obvious definitions for γ' and γ'' coming from the partition ξ.

Theorem 4.1. *There are natural algebraic maps ϕ_α and ϕ_β*

$$\begin{array}{ccc} \mathcal{M}_\alpha & & \mathcal{M}_\beta \\ & \phi_\alpha \searrow \quad \swarrow \phi_\beta & \\ & \mathcal{M}_\gamma & \end{array}$$

which are generized blow-downs along projectivizations of vector bundles over Σ_γ, where the projective fiber dimensions e_α and e_β satisfy $e_\alpha + e_\beta + 1 = \operatorname{codim} \Sigma_\gamma$.

Proof. The proof is the same as in [7], the only difference being the actual computation of the numbers e_α and e_β, which we discuss now. We assume that $E_* \sim_S E'_* \oplus E''_*$, where $[E_*] \in \Sigma_\gamma$ and $\sim_S$ denotes Seshadri equivalence (i.e. isomorphic Jordan-Hölder form). The topological type of the parabolic bundles E'_* and E''_* does not change as $[E_*]$ varies within Σ_γ. We use $(r', r''), (d', d'')$ and (m', m'') to denote the ranks, degrees, and multiplicities of (E'_*, E''_*), written as in §2. The moduli spaces $\mathcal{M}_\alpha, \mathcal{M}_\beta$, and $\mathcal{M}_\gamma$ have dimension

$$(g-1)r^2 + 1 + \frac{1}{2} \sum_{p \in D} \left(r^2 - \sum_{i=1}^{\kappa_p} m_i(p)^2 \right).$$

Using a similar formula for $\Sigma_\gamma = \mathcal{M}^{\gamma'} \times \mathcal{M}^{\gamma''}$, we find that

$$\operatorname{codim} \Sigma_\gamma = r'r''(2g-1) - 1 + \sum_{p \in D} \sum_{i=1}^{\kappa_p} m'_i(p) m''_i(p).$$

Now we claim that

$$h^0(E''^{\vee}_* \otimes E'_*) = 0 = h^0(E'^{\vee}_* \otimes E''_*).$$

This is true for any $\alpha' \in V_m$, as one of these equations is true for α, the other for β, but H^0 is constant as the weights are varied within V_m. Let $\mathcal{U}'$ and $\mathcal{U}''$ be the families parametrized by Σ_γ gotten by pulling back the universal families $\mathcal{U}^{\gamma'}$ and $\mathcal{U}^{\gamma''}$, whose existence follows from Proposition 3.2. Then the vector bundles referred to in the theorem are

$$(R^1 \pi_{\Sigma_\gamma})(\mathcal{U}''^{\vee} \otimes \mathcal{U}') \text{ and } (R^1 \pi_{\Sigma_\gamma})(\mathcal{U}'^{\vee} \otimes \mathcal{U}'').$$

The projectivizations of these bundles have dimensions

$$e_\alpha = h^1(E''^{\vee}_* \otimes E'_*) - 1 = r''d' - r'd'' + r'r''(g-1) + \chi(\mathcal{Q}) - 1, \tag{5}$$

$$e_\beta = h^1(E'^{\vee}_* \otimes E''_*) - 1 = r'd'' - r''d' + r'r''(g-1) + \chi(\mathcal{Q}') - 1, \tag{6}$$

where $\mathcal{Q}$ and $\mathcal{Q}'$ are skyscraper sheaves supported on D obtained as the quotients

$$\mathfrak{ParHom}(E''_*, E'_*) \longrightarrow \mathfrak{Hom}(E'', E') \longrightarrow \mathcal{Q},$$

$$\mathfrak{ParHom}(E'_*, E''_*) \longrightarrow \mathfrak{Hom}(E', E'') \longrightarrow \mathcal{Q}'.$$

It is a nice exercise to see

$$\chi(\mathcal{Q}) + \chi(\mathcal{Q}') = \sum_{p\in D} \left(r'r'' - \sum_{(i,j)\in S_e(p)} m_i'(p) m_j''(p) \right),$$

where $S_e(p) = \{(i,j) \mid \gamma_i'(p) = \gamma_j''(p)\}$. This shows $e_\alpha + e_\beta + 1 = \operatorname{codim} \Sigma_\gamma$. ☐

Theorem 4.2. *Suppose that $\alpha \in V_\ell$, $\beta \in V_m$, $V_\ell > V_m$, and that α and β are generic and are not separated by any hyperplanes. Then there exists a fibration $\psi : \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\beta$ with fiber a (possibly twisted) product of flag varieties and this fibration is locally trivial in the Zariski topology. In particular, $\mathcal{M}_\alpha$ is birational to the product of $\mathcal{M}_\beta$ with a product of flag varieties.*

Proof. The hypothesis $V_\ell > V_m$ just means that the flag structure degenerates as we pass from α to β. By induction, it is enough to prove the above statement when the degeneration of the flag structure is taking place at only one parabolic point. Given E_* a parabolic bundle with multiplicities m and weights α, let E_*' be the parabolic bundle with multiplicities ℓ and weights β resulting from forgetting part of the flag structure and interchanging the weights. One easily verifies that if E_* is α-stable, then E_*' is β-stable, and the existence of the morphism ψ then follows from the coarseness property of $\mathcal{M}_\beta$.

The remaining issue is to identify the fiber and to prove local triviality. For the first issue, notice that there is an inverse procedure to the forgetful map described above. Given a parabolic bundle E_*' with multiplicities ℓ and weights β, consider all parabolic bundles E_* with weights α obtained from E_*' by refining the flag stucture to one with multiplicities m and exchanging the weights. For a given E_*', the set of all such possible refinements E_* is parametrized by a flag variety.

A straightforward numerical verification shows that applying this procedure to a β-stable parabolic bundle E_*' yields an α-stable E_* for every possible refinement. It is not hard to see that the same procedure, when applied to the universal family $\mathcal{U}_*^\beta$, identifies $\mathcal{M}_\alpha$ with the total space of the flag bundle of $\mathcal{U}^\beta$ restricted to $\mathcal{M}_\beta \times \{p\}$ and the map ψ with the bundle projection. ☐

One might expect from Theorem 4.1 that the birational type of $\mathcal{M}_\alpha$ depends only on the underlying quasi-parabolic structure. This is the content of the following proposition.

Proposition 4.3. *Suppose that $g \geq 2$. Then the birational type of $\mathcal{M}_\alpha$ is independent of the choice of $\alpha \in V_m$.*

Proof. We prove the proposition by showing that $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are birational whenever $\alpha, \beta \in V_m$ are not separated by any walls (although one may lie on a wall which does not contain the other). So assume that $\alpha \in \cap_{i=1}^{n} H_{\xi_i}$ and $\beta \in \cap_{i=1}^{m} H_{\xi_i}$, where $m \geq n$. By Theorem 4.1 [10], $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are normal, projective varieties and $\dim \mathcal{M}_\alpha = \dim \mathcal{M}_\beta$, hence we only need to construct an injective morphism $\phi : \mathcal{M}_\beta^s \longrightarrow \mathcal{M}_\alpha^s$ to conclude $\mathcal{M}_\alpha$ is birational to $\mathcal{M}_\beta$. One easily verifies that every β-stable bundle is α-stable, and the existence of ϕ follows from the coarseness of $\mathcal{M}_\alpha$. □

5. Shifting and the Hecke correspondence

In this section, we introduce the notion of a shifted parabolic bundle, which is the result of changing the weights, multiplicities, and degree of E_* in a prescribed way. In some sense, shifting is a symmetry of a larger weight space, one which includes bundles of different degrees. Two applications of shifting are discussed at the end.

Shifting is most naturally described in terms of parabolic sheaves. If $\mathcal{E}$ is a locally free sheaf on X, then a *parabolic structure* on $\mathcal{E}$ consists of a weighted filtration of the form

$$\mathcal{E} = \mathcal{E}_{\alpha_1} \supset \mathcal{E}_{\alpha_2} \supset \cdots \supset \mathcal{E}_{\alpha_l} \supset \mathcal{E}_{\alpha_{l+1}} = \mathcal{E}(-D), \tag{7}$$

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < \alpha_{l+1} = 1. \tag{8}$$

We can define $\mathcal{E}_x$ for $x \in [0,1]$ by setting $\mathcal{E}_x = \mathcal{E}_{\alpha_i}$ if $\alpha_{i-1} < x \leq \alpha_i$, and then extend to $x \in \mathbb{R}$ by setting $\mathcal{E}_{x+1} = \mathcal{E}_x(-D)$. We call the resulting filtered sheaf $\mathcal{E}_*$ a parabolic sheaf and $\mathcal{E} = \mathcal{E}_0$ the underlying sheaf.

We can define parabolic subsheaves, degree, and stability for these objects, and there is a categorical equivalence between locally free parabolic sheaves and parabolic bundles. We describe this in case $D = p$, the general case being quite similar ([18], [8]).

Suppose that E_* is a parabolic bundle given by flags and weights in the fibers as in (1) and (2). Define $\mathcal{E}_*$ by setting

$$\mathcal{E}_x = \ker(E \to E_p/F_i),$$

for $\alpha_{i-1} < x < \alpha_i$. Thus $\mathcal{E}_*$ is a parabolic sheaf. Conversely, given a parabolic sheaf $\mathcal{E}_*$, the quotient $\mathcal{E}_0/\mathcal{E}_1 = \mathcal{E}/\mathcal{E}(-p)$ is a skyscraper sheaf with support p and fiber that of $\mathcal{E}$. Defining a flag in this fiber by setting $F_i = (\mathcal{E}_{\alpha_i}/\mathcal{E}_1)_p$ and associating the weight α_i, we obtain a parabolic bundle in the sense of (1) and (2).

The category of parabolic sheaves is developed in [18], where one finds for example the definitions of tensor products $\mathcal{E}_* \otimes \mathcal{E}'_*$ and duals $\mathcal{E}_*^\vee$. We use this notation freely in the calculations of §6 involving sheaf cohomology and point out that $H^i(\mathcal{E}_*) = H^i(\mathcal{E})$.

Definition 5.1. *Given a parabolic sheaf $\mathcal{E}_*$ and $\eta \in \mathbb{R}$, define the shifted parabolic sheaf $\mathcal{E}_*[\eta]_*$ by setting $\mathcal{E}_*[\eta]_x = \mathcal{E}_{x+\eta}$.*

Remark. The above operation can be refined in case $D = p_1 + \cdots + p_n$. If $\eta = (\eta_1, \ldots, \eta_n)$, then one can shift $\mathcal{E}_*$ by η_i at each $p_i \in D$ ([18], [7]).

It is not difficult to verify that $\mathcal{E}_*[\eta]_*$ is (semi)stable if and only if $\mathcal{E}_*$ is (semi)stable, and it follows that this defines an isomorphism between the associated moduli spaces of parabolic bundles.

We can easily describe the parabolic structure on the shifted bundle $\mathcal{E}'_* = \mathcal{E}_*[\eta]_*$ in case $0 < \eta \leq 1$ and $D = p$. Let E'_* denote the parabolic bundle associated to $\mathcal{E}'_*$. If i is the integer with $\alpha_i < \eta \leq \alpha_{i+1}$, then the weights of E'_* are given by

$$\alpha'_j = \begin{cases} \alpha_{j+i} - \eta & \text{for } j = 1, \ldots, r-i, \\ 1 + \alpha_{j-r+i} - \eta & \text{for } j = r-i+1, \ldots, r. \end{cases} \tag{9}$$

The quasi-parabolic structure of E'_* has multiplicities m' given by a cyclic permutation of m, i.e. $m' = (m_{i+1}, \ldots, m_\kappa, m_1, \ldots, m_i)$. Although $\mathcal{E}'$ is a subsheaf of $\mathcal{E}$, E' is *not* a subbundle of E, so one must appeal to sheaf theory in order to define the flag in E'_p. This is a simple exercise in tracing through the equivalence between locally free parabolic sheafs and parabolic bundles given above.

There are two interesting applications of shifting we discuss now. The first is the Hecke correspondence. Using $\mathcal{M}_{r,d}$ to denote the moduli space of semistable bundles of rank r and degree d, the Hecke correspondence gives a means of comparing $\mathcal{M}_{r,d}$ and $\mathcal{M}_{r,d'}$ through the use of parabolic bundles. For $r = 2$, this was observed in a remark at the end of [10].

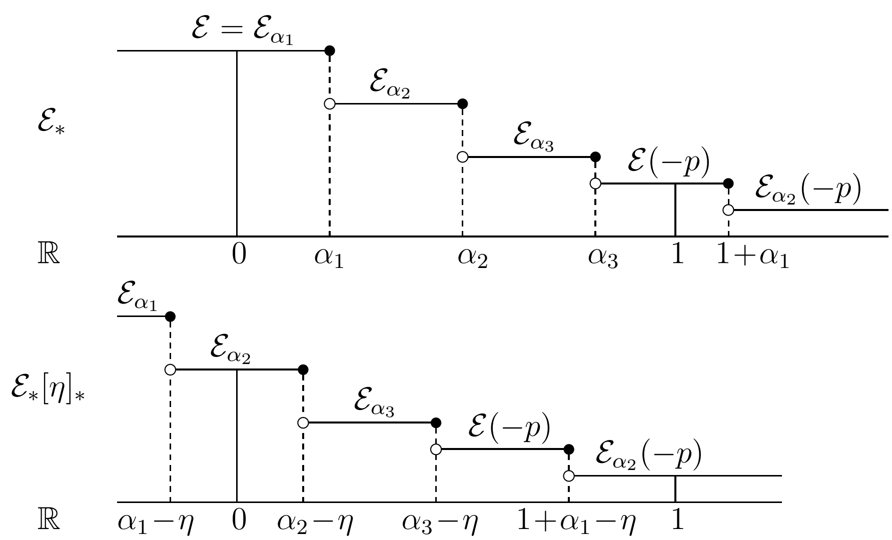

FIGURE 1. The parabolic sheaf $\mathcal{E}_*$ shifted by η with $\alpha_1 < \eta < \alpha_2$.

To start, define $\epsilon_+(d,r), \epsilon_-(d,r)$, and $\epsilon(d,r)$ for $d,r \in \mathbb{Z}$ with $r > 0$ by

$$\begin{aligned}\epsilon_\pm(d,r) &= \inf\{\pm(\tfrac{d}{r} - \tfrac{d'}{r'}) \mid d', r' \in \mathbb{Z},\ 1 \leq r' < r, \text{ and } \pm(\tfrac{d}{r} - \tfrac{d'}{r'}) > 0\} \\ \epsilon(d,r) &= \min\{\epsilon_\pm(d,k) \mid k = 1, \dots, r\}.\end{aligned}$$

It is easy to see that $\epsilon_\pm(d,k) > 0$ for all k, thus $\epsilon(d,r) > 0$ as well.

Suppose that E is a bundle over X of degree d and rank r and suppose further that E' is a proper subbundle. If $\mu(E') < \mu(E)$, then $\mu(E) - \mu(E') \geq \epsilon_+(d,r)$. Similarly, if $\mu(E') > \mu(E)$, then $\mu(E') - \mu(E) \geq \epsilon_-(d,r)$.

Proposition 5.2. *Suppose that E_* satisfies* $\displaystyle\sum_{p\in D}\sum_{i=1}^{\kappa_p} m_i(p)\alpha_i(p) < \epsilon(d,r)/2.$

(i) *If E is stable as a regular bundle, then E_* is parabolic stable.*

(ii) *If E_* is parabolic stable, then E is semistable as a regular bundle.*

Proof. (i) If E'_* is a proper parabolic subbundle of E_*, then

$$\mu(E'_*) \leq \mu(E') + \epsilon(d,r)/2 < \mu(E') + \epsilon_+(d,r) \leq \mu(E) < \mu(E_*),$$

thus E_* is parabolic stable.

(ii) If E' is a subbundle of E, then

$$\mu(E') \leq \mu(E'_*) < \mu(E_*) < \mu(E) + \epsilon(d,r)/2 < \mu(E) + \epsilon_-(d,r),$$

hence $\mu(E') \leq \mu(E)$ and E is semistable. □

We thus get a morphism $\mathcal{M}_\alpha \longrightarrow \mathcal{M}_{r,d}$ which is the map of Theorem 4.2 in case $(r,d) = 1$. By choosing the weights and quasi-parabolic structure correctly, we can fit $\mathcal{M}_{r,d}$ and $\mathcal{M}_{r,d-1}$ into a chain diagram of maps as follows. Let $D = p$ and $m = (1, \dots, 1)$, and choose weights $\alpha = (\alpha_1, \dots, \alpha_r)$ with $\alpha_1 + \cdots + \alpha_r < \epsilon(r,d)/2$. Suppose $\alpha_1 < \eta < \alpha_2$ and set E'_* to be the parabolic bundle E_* shifted by η. Notice that E'_* has degree $d-1$, multiplicities $m' = (1, \dots, 1)$, and weights $\alpha' = (\alpha_2 - \eta, \dots, \alpha_r - \eta, 1 - \eta + \alpha_1)$. If $\beta' \in V_{m'}$ is generic with $\beta'_1 + \cdots + \beta'_r < \epsilon(r,d)/2$, then we can connect α' to β' in $V_{m'}$ by a line passing through a finite number of hyperplanes $H_{\xi^1}, \dots, H_{\xi^n}$, all of the form to which Theorem 4.1 applies. Choose weights α^i in the intermediate chambers and $\gamma^i \in H_{\xi^i}$ for $i = 1, \dots, n$ with $\alpha^n = \beta'$. Applying Theorem 4.1 each time we cross a hyperplane, we get the following diagram:

$$\begin{array}{ccccccc} \mathcal{M}_\alpha \cong \mathcal{M}_{\alpha'} & & & \mathcal{M}_{\alpha^1} & & & \mathcal{M}_{\beta'} \\ \psi \downarrow & \searrow & \swarrow & & \searrow & \swarrow & \downarrow \psi' \\ \mathcal{M}_{r,d} & & \mathcal{M}_{\gamma^1} & & \cdots & & \mathcal{M}_{r,d-1} \end{array} \qquad (10)$$

where, by the above proposition, the vertical maps ψ and ψ' have fibers the (full) flag variety over $\mathcal{M}^s_{r,d}$ and $\mathcal{M}^s_{r,d-1}$, respectively. By Theorem 4.2, ψ is a fibration which is locally trivial in the Zariski topology provided $(r,d) = 1$, and the same follows for ψ' if $(r, d-1) = 1$.

The second application of shifting is to extend the results of [7] to a case which is natural from the point of view of representations of Fuchsian groups but less natural from the point of view of parabolic bundles. Assume for simplicity that $\mu(E_*) = 0$ and $D = p$. Thus, $\deg E = -k$ for some $0 \le k < r$, and the relevant weight space is

$$W_k = \{(\alpha_1, \dots, \alpha_r) \in \Delta^r \mid \alpha_1 + \cdots + \alpha_r = k\}.$$

Consider the union $\widetilde{W} = \bigcup_{k=0}^{r-1} W_k$, where we identify

$$\partial_0 W_k = \{\gamma \in W_k \mid \gamma_1 = 0\}$$

with its companion set

$$\partial_1 \overline{W}_{k+1} = \{\overline{\gamma} \in \overline{W}_{k+1} \mid \overline{\gamma}_r = 1\}$$

via the identification

$$\partial_0 W_k \ni \gamma = (0, \gamma_2, \dots, \gamma_n) \sim (\gamma_2, \dots, \gamma_n, 1) = \overline{\gamma} \in \partial_1 \overline{W}_{k+1}. \tag{11}$$

One can think of this set $\widetilde{W}$ as the space of all weights modulo shifting[2], which in this case is just the quotient $\mathrm{SU}(r)/Ad$ and which can be naturally identified with the standard $r-1$ simplex. From this point of view $\partial_0 W_k$ is an interior hyperplane of $\widetilde{W}$ because it satisfies condition (3).

However, Theorem 4.1 does not obviously carry over to this case because points in W_k and W_{k+1} are weights on parabolic bundles of different degrees. Given a parabolic bundle of degree $-k$, what is needed is a canonical procedure to construct a parabolic bundle of degree $-(k+1)$. This is precisely what is provided by the shifting operation. Thought of in terms of $\widetilde{W}$, the following theorem extends Theorem 4.1 to the case where $H_\xi = \partial_0 W$.

We use the notation $\mathcal{M}_\alpha(k, m)$ for the moduli space when E_* has degree $-k$, multiplicities m, and weights α.

Theorem 5.3. *Suppose that $\gamma \in \partial_0 W_k \cap V_m$ does not lie on any other hyperplanes and that $\alpha \in W_k \cap V_m$ is a generic weight near to γ. Choose $\eta \in \mathbb{R}$ with $0 < \eta < \gamma_{m_1+1}$. Define $\overline{\gamma} \in \partial_1 \overline{W}_{k+1}$ as in (11). Let E'_* be E_* shifted by η, and denote the multiplicities of E'_* by m'.*

[2]Because every bundle can be shifted so that $\mu(E_*) = 0$.

Set $k' = -\deg E' = k + m_1$. Let $\beta \in W_{k'} \cap V_{m'}$ be generic near $\overline{\gamma}$. Then there are projective algebraic maps ϕ_α and ϕ_β

$$\begin{array}{ccccc} \mathcal{M}_\alpha(m,k) & & & & \mathcal{M}_\beta(m',k') \\ & \phi_\alpha\searrow & & \swarrow\phi_\beta & \\ & & \mathcal{M}_\gamma(m,k) & & \end{array}$$

satisfying the conclusion of Theorem 4.1.

Proof. By the choice of α, β and η, we see that $\alpha_{m_1} < \eta < \alpha_{m_1+1},\ \eta < \beta_1$ and $\eta < \gamma_{m_1+1}$. Consequently, the shifting operation defines the following isomorphisms:

$$\begin{aligned} \mathcal{M}_\alpha(m,k) &\cong \mathcal{M}_{\alpha'}(m',k'), \\ \mathcal{M}_\beta(m',k') &\cong \mathcal{M}_{\beta'}(m',k'), \\ \mathcal{M}_\gamma(m,k) &\cong \mathcal{M}_{\gamma'}(m',k'), \end{aligned}$$

where $\alpha', \beta', \gamma' \in V_{m'}$ are defined from α, β, γ as in (9). Now Theorem 4.1 applies to the shifted moduli spaces to prove the theorem. One can calculate e_α and e_β by applying formulas (5) and (6) to α', β' and γ'. $\square$

Remark. Theorem 5.3 solves a problem mentioned at the end of [7] and extends the wall-crossing formula for knot invariants introduced in [6].

6. Rationality of moduli spaces of parabolic bundles

Let L be a holomorphic line bundle over a curve X of genus $g \geq 2$. Denote by

(i) $\mathcal{M}_{r,L}$ the moduli space of semistable bundles E of rank r with $\det E = L$, and by

(ii) $\mathcal{M}_{\alpha,L}$ the moduli space of parabolic bundles E_* with weights α and $\det E = L$.

The main results of §4 hold for the moduli spaces with fixed determinant with no essential difference. In view of Theorem 4.2, the goal is therefore to prove rationality with the coarsest possible choice of flag structure. At one extreme, we have the trivial flag, whose moduli space is exactly $\mathcal{M}_{r,L}$. Proposition 2 of [12] implies that $\mathcal{M}_{r,L}$ is rational if $\deg L = \pm 1 \mod (r)$, and then Theorem 4.2 and Proposition 4.3 imply that $\mathcal{M}_{\alpha,L}$ is also rational for any $\alpha \in V_m$ provided $\deg L = \pm 1 \mod (r)$.

Theorem 6.1. *If $m(p) = (1, \ldots, 1)$ for some $p \in D$, then $\mathcal{M}_{\alpha,L}$ is rational for all $\alpha \in V_m$.*

Proof. First, use Theorem 4.2 to reduce to the case $D = p$ by forgetting all the other flag structures. If E'_* denotes the bundle obtained by shifting E_* by some η with $\alpha_1 < \eta < \alpha_2$, then $\det E' = L' = L(-p)$. It follows that shifting by η defines an isomorphism from $\mathcal{M}_{\alpha,L}$

to $\mathcal{M}_{\alpha',L'}$. Repeated application of shifting puts us in the case $\deg L = 1 \mod (r)$, and then Newstead's theorem and Theorem 4.2 imply that $\mathcal{M}_{\alpha,L}$ is rational. ☐

The above argument works in slightly more generality. We can always shift our bundle to be any of the $\mathcal{E}_x$ appearing in the filtration (7) and illustrated in Figure 1. Thus, whenever one of these terms in the filtration is of a degree to which Newstead's theorem applies, the corresponding moduli space of parabolic bundles is rational.

The next theorem is a considerable strengthening of the previous one.

Theorem 6.2. *If $m_i(p) = 1$ for some $p \in D$ and some $1 \leq i \leq \kappa_p$, then $\mathcal{M}_{\alpha,L}$ is rational for all $\alpha \in V_m$.*

Before delving into the proof of this theorem, we mention some interesting consequences. Recall first the following definition.

Definition 6.3. *A variety V is stably rational of level k if $V \times \mathbb{P}^k$ is rational. The level is the smallest integer k with this property.*

The following result, with a weaker bound on the level, was proved in [2].

Corollary 6.4. *For $(r, d) = 1$, $\mathcal{M}_{r,L}$ is stably rational with level $k \leq r - 1$.*

Proof. Theorem 6.2 implies that $\mathcal{M}_{\alpha,L}$ is rational, where $m(p) = (r - 1, 1)$, and Theorem 4.2 shows that $\mathcal{M}_{\alpha,L}$ is birational to $\mathcal{M}_{r,L} \times \mathbb{P}^{r-1}$, which proves the corollary. ☐

We now apply this last result to Conjecture 1.1.

Corollary 6.5. *Suppose $(r, d) = 1$. By tensoring with a line bundle, we can assume that $0 < d < r$. If either $(g, d) = 1$ or $(g, r - d) = 1$, then $\mathcal{M}_{r,L}$ is rational.*

Proof. Suppose first that $(g, r-d) = 1$. Let L be a line bundle of degree $r(g-1)+d$. Then Newstead's construction applies and proves that $\mathcal{M}_{r,L}$ is birational to $\mathcal{M}_{r-d,L} \times \mathbb{P}^{\chi}$, where $\chi = (g - 1)(r^2 - (r - d)^2)$. But the above corollary implies that $\mathcal{M}_{r-d,L}$ is stably rational with level $k \leq r - d - 1 \leq \chi$, hence $\mathcal{M}_{r,L}$ is rational.

The case $(g, d) = 1$ follows by the same argument after applying duality, which interchanges (r, d) and $(r, r - d)$. ☐

Remark. Conjecture 1.1 was previously known [12] in the following three cases:

(i) $d = \pm 1 \mod (r)$,

(ii) $(r, d) = 1$ and g a prime power, and

(iii) $(r, d) = 1$ and the two smallest distinct primes factors of g have sum greater than r.

Corollary 6.5 applies in each case. More importantly, it applies in many cases not covered by (i), (ii) or (iii). In fact, for a given r and d with $(r,d)=1$, one can easily list those g for which the conjecture remains open. For example, if $r=110$ and $d=43$, then Corollary 6.5 applies as long as g is not a multiple of $d\cdot(r-d)=43\cdot 67=2881$.

Proof of Theorem. Set $d=\deg L$. The theorem is clearly true for $r=1$ and follows from Theorem 6.1 for $r=2$, so assume $r>2$. Notice that by tensoring with a line bundle, we can suppose

$$r(g-1)<d\leq rg.$$

By Theorem 4.2, we can again assume that $D=p$, and by shifting and another application of Theorem 4.2, if necessary, we can arrange it so that $m(p)=(r-1,1)$. Write

$$\alpha=\alpha(p)=(\overbrace{\alpha_1,\cdots,\alpha_1}^{r-1},\alpha_2).$$

Proposition 3.2 implies that V_m contains a generic weight and that $\mathcal{M}_{\alpha,L}$ parametrizes a universal family $\mathcal{U}_*^\alpha$. By Proposition 4.3, the birational type of $\mathcal{M}_{\alpha,L}$ is independent of choice of compatible weights, so we can assume that the weights are small enough to satisfy the hypothesis of Proposition 5.2 (this comes up at various technical points in the argument, e.g. the proof of Claim 6.6).

Consider the following two cases.

Case I: $d=rg$. Choose η with $\alpha_1<\eta<\alpha_2$, and let $E'_*=E_*[\eta]_*$. Denote the weights of E'_* by α' as in (9). If $\det E=L$, then $\det E'=L'=L(-(r-1)p)$ has degree $d'=d-(r-1)$. Since $d'=1 \mod (r)$, Proposition 2 of [12] implies that $\mathcal{M}_{r,L'}$ is rational, and Theorem 4.2 then implies that $\mathcal{M}_{\alpha',L'}$ is also rational. Rationality of $\mathcal{M}_{\alpha,L}$ now follows from the isomorphism of the moduli spaces $\mathcal{M}_{\alpha,L}\cong\mathcal{M}_{\alpha',L'}$ defined by shifting by η.

Case II: $r(g-1)<d<rg$. The idea is to use induction to construct a nonempty, Zariski-open subset $\mathcal{M}$ of affine space of dimension $(r^2-1)(g-1)+r-1$ $(=\dim\mathcal{M}_{\alpha,L})$ and a family of stable parabolic bundles $\mathcal{U}_*$ parametrized by $\mathcal{M}$ with $\det\mathcal{U}_{\xi,*}=L$ for all $\xi\in\mathcal{M}$. The universal property of $\mathcal{U}_*^\alpha$ then gives a map $\psi_{\mathcal{U}_*}:\mathcal{M}\longrightarrow\mathcal{M}_{\alpha,L}$. If, in addition, we have $\mathcal{U}_{\xi_1,*}\cong\mathcal{U}_{\xi_2,*}\ \Leftrightarrow\ \xi_1=\xi_2$, then $\psi_{\mathcal{U}_*}$ is injective and rationality of $\mathcal{M}_{\alpha,L}$ follows from that of $\mathcal{M}$ and the dimension condition.

Set $r'=rg-d$, $r''=r-r'$ and $\alpha'=(\overbrace{\alpha_1,\cdots,\alpha_1}^{r'-1},\alpha_2)$. Assume that both α and α' are generic. Let $\mathcal{U}_*^{\alpha'}$ be the universal family parametrized by $\mathcal{M}_{\alpha',L}$ and $I_*=\mathcal{O}_X[\alpha_1]_*$ be the trivial parabolic line bundle with weight α_1. If $e' = [E'_*] \in \mathcal{M}_{\alpha',L}$, then because $E'^{\vee}_* \otimes I_*$ is a stable parabolic bundle of negative parabolic degree, $h^0(E'^{\vee}_* \otimes I_*) = 0$ and

$$n \stackrel{\text{def}}{=} h^1(E'^{\vee}_* \otimes I_*) = (2r' + r'')(g-1) + r'' + 1 \tag{12}$$

is independent of e'. Since $\mathcal{U}^{\alpha'}_{e',*} \cong E'_*$, it follows that

$$(R^1\pi_{\mathcal{M}_{\alpha',L}})((\mathcal{U}^{\alpha'}_*)^{\vee} \otimes \pi_X^*(I_*))$$

is locally free. The associated vector bundle $V \xrightarrow{\pi} \mathcal{M}_{\alpha',L}$ has rank n and fiber over e' naturally isomorphic to $H^1(E'^{\vee}_* \otimes I_*)$.

Let $\mathcal{U}'_* = (\pi^{r''} \times 1_X)^*(\mathcal{U}^{\alpha'}_*)$ be the pullback family and $\mathcal{I}^{\oplus r''}_* = \pi_X^* I^{\oplus r''}_*$ the trivial family, where $\pi^{r''} : V^{\oplus r''} \longrightarrow \mathcal{M}_{al',L}$. There is an extension

$$0 \longrightarrow \mathcal{I}^{\oplus r''}_* \longrightarrow \mathcal{U}_* \longrightarrow \mathcal{U}'_* \longrightarrow 0 \tag{13}$$

of families over $V^{\oplus r''} \times X$, such that, for $\xi \in V^{\oplus r''}_{e'}$, $\mathcal{U}_{\xi,*}$ is the parabolic bundle E^{ξ}_* described as the short exact sequence

$$0 \longrightarrow I^{\oplus r''}_* \longrightarrow E^{\xi}_* \longrightarrow E'_* \longrightarrow 0 \tag{14}$$

corresponding to the extension class $\xi \in H^1(E'^{\vee}_* \otimes I^{\oplus r''}_*)$.

Using stability of E'_* and triviality of $I^{\oplus r''}_*$, it follows that

$$\mathrm{Aut}(E'_*) \times \mathrm{Aut}(I^{\oplus r''}_*) \cong \mathbb{C}^* \times \mathrm{GL}(r'', \mathbb{C}).$$

This group acts naturally as fiber-preserving maps on the bundle $V^{\oplus r''}$ since

$$V^{\oplus r''}_{e'} \cong H^1(E'^{\vee}_* \otimes I^{\oplus r''}_*) = H^1(E'^{\vee}_* \otimes I_*)^{\oplus r''},$$

and two extension classes ξ_1 and ξ_2 in the same orbit have associated bundles E^{ξ_1} and E^{ξ_2} which are isomorphic. We can ignore the $\mathbb{C}^*$ action here because $(z,1)\cdot\xi = (1,z)\cdot\xi$ for $z \in \mathbb{C}^*$ and $\xi \in V^{\oplus r''}$.

Using the inductive hypothesis and local triviality of V, we can choose a nonempty Zariski-open subset $\mathcal{M}'$ of $\mathcal{M}_{\alpha',L}$ isomorphic to a Zariski-open subset of affine space of dimension $(r'^2 - 1)(g-1) + r' - 1$ such that $V|_{\mathcal{M}'} \cong \mathcal{M}' \times H^1(E'^{\vee}_* \otimes I_*)$ (E'_* is fixed). Lemma 2 of [12] applies here and produces a Zariski-open subspace $\mathcal{M}' \times W$ of $V^{\oplus r''}|_{\mathcal{M}'}$ invariant under the group action, and an affine subspace $A \subset W$ so that every orbit in W intersects A precisely once. In fact, A can be chosen as a Zariski-open subset of the Grassmannian $G(r'', n)$. In any case, it should be clear that A has dimension $r''(n - r'')$.

Using equation (12) and the fact that $r' + r'' = r$, we see that $\mathcal{M}' \times A$ is a Zariski-open subset of affine space of dimension

$$\begin{aligned} \dim \mathcal{M}' \times A &= (r'^2 - 1)(g-1) + r' - 1 + r''(n - r'') \\ &= (r'^2 - 1)(g-1) + r' - 1 + r''((2r' + r'')(g-1) + 1) \\ &= (r^2 - 1)(g-1) + r - 1. \end{aligned}$$

Let $\mathcal{M}$ be the subset of $V^{\oplus r''}$ defined by

$$\mathcal{M} = \{\xi \in \mathcal{M}' \times A \mid H^1(\mathcal{U}_{\xi,*}) = 0\},$$

and consider the bundle $\mathcal{U}_*$ restricted to $\mathcal{M}$, which we continue to denote $\mathcal{U}_*$. For $\xi \in V^{\oplus r''}$, let $E^\xi_* = \mathcal{U}_{\xi,*}$. Clearly E^ξ_* is a parabolic bundle with weights α and determinant L, thus $\mathcal{M}$ parametrizes a family of parabolic bundles. By the upper semi-continuity theorem, $\mathcal{M}$ is Zariski-open in $\mathcal{M}' \times A$.

We claim that $\mathcal{M}$ is nonempty. Fix $e' = [E'_*] \in \mathcal{M}'$ and consider the set

$$N = \{\xi \in H^1(E'^{\vee}_* \otimes I^{\oplus r''}_*) \mid h^1(E^\xi_*) = 0\}.$$

If $N \cap A \neq \emptyset$, then $\mathcal{M}$ is nonempty. Clearly, N is invariant under the action of $\mathrm{GL}(r'', \mathbb{C})$, so it is enough to show $N \cap W \neq \emptyset$. There is a natural map

$$\delta : H^1(E'^{\vee}_* \otimes I^{\oplus r''}_*) \times H^0(E'_*) \longrightarrow H^1(I^{\oplus r''}_*)$$

with $\delta_\xi = \delta(\xi, \cdot) : H^0(E'_*) \longrightarrow H^1(I^{\oplus r''}_*)$ the coboundary map of the long exact sequence in homology of (14). Now $H^0(E'_*) = H^0(E')$, and since $\alpha_1 + (r'-1)\alpha_2 < \epsilon(r,d)/2$, by Proposition 5.2, E' is semistable as a non-parabolic bundle. Serre duality implies that $h^1(E') = h^0(E'^{\vee} \otimes K)$, and we compute

$$\begin{aligned} \deg(E'^{\vee} \otimes K) &= -d + r'(1-g) \\ &\leq (r + r')(1-g) - r'', \end{aligned}$$

which is negative since $r'' \geq 1$ and $g \geq 2$. This implies that $h^1(E'_*) = 0$, and Riemann-Roch implies that $h^0(E'_*) = r''g$. Because $h^1(I^{\oplus r''}_*) = r''g$, we see that

$$\xi \in N \iff H^1(E^\xi_*) = 0 \iff \delta_\xi \text{ is an isomorphism.}$$

But δ is obviously onto and $\dim(\ker \delta) = r''n$. The set N has complement

$$N^c = \{\xi \in H^1(E'^{\vee}_* \otimes I^{\oplus r''}_*) \mid \delta(\xi, s) = 0 \text{ for some } 0 \neq s \in H^0(I^{\oplus r''}_*)\}.$$

But $\delta(\xi, s) = 0 \Rightarrow \delta(\xi, zs) = 0$ for all $z \in \mathbb{C}$, which shows that the map $\ker \delta \longrightarrow N^c$ has fibers of dimension ≥ 1. Hence $\dim N^c \leq \dim(\ker \delta) - 1 < r''n$, and we see that N is nonempty and Zariski-open. Thus $N \cap W \neq \emptyset$ and it follows that $\mathcal{M}$ is nonempty.

We now prove that $\mathcal{M}$ parametrizes a family of stable parabolic bundles, using again the inequality $(r-1)\alpha_1 + \alpha_2 < \epsilon(r,d)/2$ and Proposition 5.2.

Claim 6.6. (i) *E^{ξ}_* is stable for all $\xi \in \mathcal{M}$.*
(ii) *$E^{\xi_1}_* \cong E^{\xi_2}_* \Longleftrightarrow \mathrm{GL}(r'', \mathbb{C}) \cdot \xi_1 = \mathrm{GL}(r'', \mathbb{C}) \cdot \xi_2$ for all $\xi_1, \xi_2 \in \mathcal{M}$.*

Proof. (i) Suppose to the contrary that E^{ξ}_* is not parabolic stable for some $\xi \in \mathcal{M}$. Let G_* be a rank s parabolic subbundle of E^{ξ}_* with $\mu(G_*) > \mu(E^{\xi}_*)$. Then $\mu(G) \geq \mu(E^{\xi})$, since otherwise

$$\mu(G_*) < \mu(G) + \epsilon(d,r)/2 < \mu(E^{\xi}) < \mu(E^{\xi}_*).$$

As in the argument of Lemma 6 of Newstead, the map $G \longrightarrow E'$ has a factorization as $G \to G^1 \to G^2 \to E'$ and the arguments there give the following inequalities:

$$\deg(G^2) \geq \deg(G) \geq \frac{sd}{r}, \tag{15}$$

$$\mathrm{rank}(G^2) \leq \mathrm{rank}(G) - h^0(G) \leq \frac{sr'}{r}. \tag{16}$$

These imply that $\mu(G^2) - \mu(E') \geq 0$. But E'_* is parabolic stable, so by Proposition 5.2, E' is semistable and $\mu(G^2) = \mu(E')$. Thus, we must have equalities in equations (15) and (16), in particular $\mu(G) = \mu(E^{\xi})$. But since $\mu(G_*) > \mu(E^{\xi}_*)$, we see that G_* must inherit the weight α_2, which implies that G^2_* also inherits α_2, and it now follows that

$$\mu(G^2_*) - \mu(E'_*) = \frac{(s_2-1)\alpha_1 + \alpha_2}{s_2} - \frac{(r'-1)\alpha_1 + \alpha_2}{r'} > 0,$$

where $s_2 = \mathrm{rank}\, G^2 < r'$. This contradicts the parabolic stability of E'_* and completes the proof of part (i).

(ii) Since $\Leftarrow$ is true independent of the vanishing of H^1, we only prove $\Rightarrow$. Suppose $E^{\xi_1}_* \cong E^{\xi_2}_*$ and set $\pi_X(E^{\xi_i}_*) = e'_i = [E^{i'}_*] \in \mathcal{M}_{\alpha',L}$. Notice that $h^1(E^{\xi_i}_*) = 0$, and so $h^0(E^{\xi_i}_*) = \chi(E^{\xi_i}_*) = r''$. It follows that every holomorphic section of $E^{\xi_i}_*$ has its image contained in $I^{\oplus r''}_*$. Hence any isomorphism $\varphi : E^{\xi_1}_* \longrightarrow E^{\xi_2}_*$ defines a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^{\oplus r''}_* & \longrightarrow & E^{\xi_1}_* & \longrightarrow & E^{1'}_* & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \varphi''} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \varphi'} & & \\
0 & \longrightarrow & I^{\oplus r''}_* & \longrightarrow & E^{\xi_2}_* & \longrightarrow & E^{2'}_* & \longrightarrow & 0
\end{array}$$

where both φ' and φ'' are isomorphisms, and so $\xi_2 = (\varphi' \times \varphi'') \cdot \xi_1$. □

Part (i) of the claim and the universal property of $\mathcal{U}^{\alpha}_*$ gives a map $\mathcal{M} \xrightarrow{\psi} \mathcal{M}_{\alpha,L}$, which is injective by part (ii). Since $\mathcal{M}$ is nonempty, $\dim \mathcal{M} = \dim \mathcal{M}_{\alpha,L}$, so rationality of $\mathcal{M}_{\alpha,L}$ follows from that of $\mathcal{M}$. This concludes the proof in Case II. □

Remark. We had originally hoped to prove rationality of $\mathcal{M}_{\alpha,L}$ with the weaker hypothesis that α is generic, but the argument does not hold in this generality. For consider the case $D = p$. By tensoring with a line bundle and shifting, we can assume that

$$r(g-1) < d \leq r(g-1) + m_1.$$

Hence, the subbundle split off in the induction is again a sum of parabolic line bundles with the same weights. The difficulty is in proving that the quotient E'_* has *generic* weights α'.

Proposition 3.2 implies that E'_* admits a generic weight if and only if the elements of the set $\{d, m'_i(p)\}$ greatest common divisor equal to one. The statement

$$(d, m_1, \ldots, m_\kappa) = 1 \Rightarrow (d, m'_1, \ldots, m'_\kappa) = 1,$$

which is what we would need to prove here, is unfortunately false (notice that $m'_1 = m_1 - d + r(g-1)$ and $m'_i = m_i$ otherwise).

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