Linear discrete-time models

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Univariate (one-dimensional), discrete-time, deterministic model: $N(t+1) = f(N(t))$. (Typically, state variable is continuous.) (Units of time, stock? When does discrete time make sense?)

Geometric growth (decay)

Simplest possible model. $f(N) = RN$ (sometimes stated as $f(N) = (1+r)N$). Solve recursion analytically. Now we know everything about the dynamics. Suppose $N(0) > 0$, $R > 0$ (note: the R language indexes vectors starting from 1). (What happens if $N(0) < 0$? model of debt?)

What happens if $R < 0$?

(Even this ridiculously simple rule — or generalizations of it — is the basis of serious modeling in conservation biology.)

The limiting set of points as $t \to \infty$ is called an *attractor*. If $R < 1$, $N \to 0$ but $N = 0$ only in the limit, unless it starts there. A value of N such that $f(N^*) = N^*$ is called an *equilibrium* (or a *fixed point*).

Stability: what happens for perturbations in the neighborhood of the fixed point? Consider displacing the population away from N^* by δ, where $\delta \ll 1$; what happens?

$$f(N^* + \delta) = f(N^*) + \delta f'(N)|_{N=N^*} + \delta^2/2 f''(N)|_{N=N^*} + \ldots$$

Therefore the deviation $\delta \to \delta f'(N^*)$. **If $|g| \equiv |f'(N^*)| < 1$ then the deviation from the equilibrium decreases geometrically with time**: *stable* fixed point. Behavior for $g < (-1)$, $-1 < g < 0$, $0 < g < 1$, $g > 1 \ldots$

$N = 0$ is always an equilibrium, stable iff $|R| < 1$.

Affine models

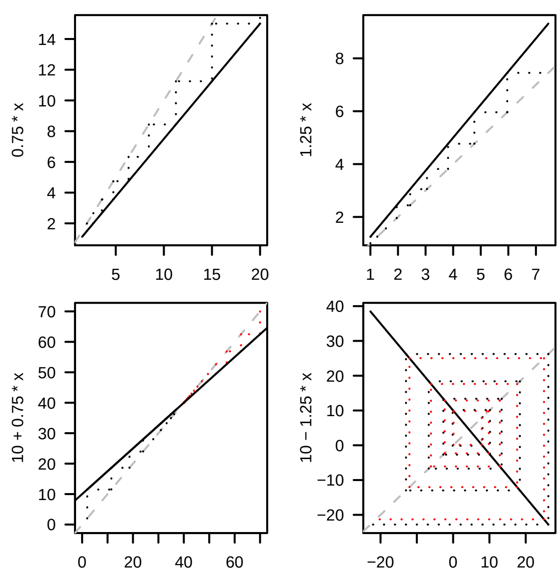
Now suppose (as in the example in the book) we are adding or subtracting a fixed amount per time step: $N(t+1) = a + bN(t)$. As before we can work out the recursion.

Summing the series for t steps gives $a(1-b^t)/(1-b) + b^t N$; the limit is $a/(1-b) + \lim_{t\to\infty} b^t N(0)$. If $|b| > 1$ this is a bit boring. If $|b| < 1$ we get a stable equilibrium at $a/(1-b)$. (For $b < 0$ ("bucket model"): a is the supply rate, $1/(1-b)$ is the average *residence time*.)

Useful component for larger models. (Autoregressive model in time series analysis; sometimes used as the bottom level in food chain modeling.)

In general, things are not so simple. The general approach is (1) solve for equilibria (directly); (2) evaluate stability of equilibria; (3) possibly evaluate for small N (near system boundaries); (4) if possible solve for time-dependent solution.

Graphical approaches: cobwebbing



Multiple lags

What if $N(t+1)$ depends on previous time steps $N(t-1)$ etc. as well as $N(t)$? Homogeneous linear equations: $\sum_{i=0}^{m} a_i N(t-i) = 0$. Plug in $N(t) = C\lambda^t$. Solve characteristic equation . . . get a linear combination of geometric growth/decay, $\sum C_i \lambda_i^n$: largest *eigenvalue* dominates long-term behavior.