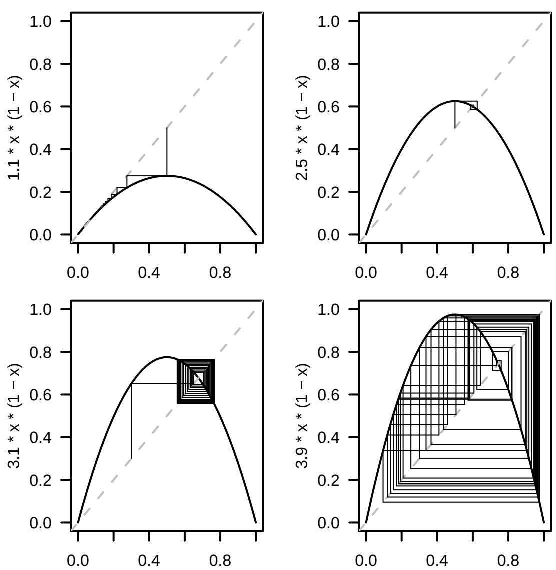
Multi-state and nonlinear discrete-time models

Logistic model

Impose bounds on an otherwise ridiculous growth process. (Mechanistic/phenomenological description.) $N(t) = N + rN(1 - N/K)$; can set $K = 1$ (*non-dimensionalization*). Get expected results for $R < 1$, R small (< 2). (Equilibrium?)



Alternative parameterizations

An ecologist or other normal person might choose to parameterize the discrete logistic model as above. A mathematician would choose $x(t+1) = Rx(1-x)$. The mathematician has chosen $R = r/K \rightarrow K = 1 - 1/R$. Mathematically equivalent parameterizations often have quite different meanings (or statistical properties), as well as cultural connotations. Get used to it.

More nonlinear models

Other 1-D discrete nonlinear models: *Ricker* model ($N = rNe^{-bN}$); population genetics; approximations of continuous models. Epidemic models (SI) (equivalent to discrete logistic).

Graphical approaches, continued: *Allee effects*. Bistability, multiple stable states.

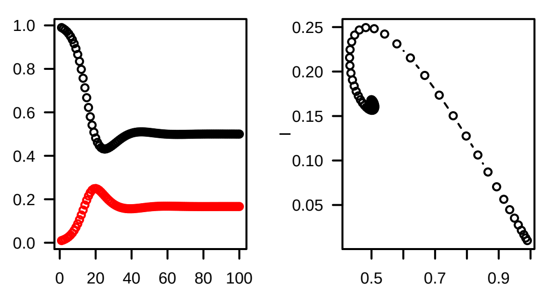
Linear multi-state/multi-lag models

Back to linear models (briefly). Juvenile/adult model: fractions $\{s_J, s_A\}$ of juveniles and adults survive; adults have f offspring each (on average); surviving juveniles become adults. So $A(t+1) = s_A A(t) + s_J J(t)$, $J(t+1) = fA(t)$ *or* $A(t+1) = s_A A(t) + s_J f A(t-1)$. Could try to solve by recursion. Or guess that the solutions are of the form $C\lambda^t$. Plug in: $C\lambda^{t+1} = s_A C\lambda^t + s_J f C\lambda^{t-1}$, or $\lambda^2 - s_A\lambda - s_J f = 0$ (*characteristic equation*). $\lambda = 1/2(s_A \pm \sqrt{s_A^2 - 4s_J f})$. CE is the same as if we wrote it in the multi-state, matrix form: $\mathbf{N} = \mathbf{MN}$ where $\mathbf{M} = \begin{pmatrix} 0 & f \\ s_J & s_A \end{pmatrix}$; then the solutions of the CE are the *eigenvalues* of $\mathbf{M}$ (i.e. values of λ s.t. $\mathbf{Mv} = \lambda\mathbf{v}$ for some *eigenvector* $\mathbf{v}$). Solution is a mixture: $A(t) = C_+\lambda_+^t + C_-\lambda_-^t$.

What can we say about $\lambda_\pm$? Can put in values (e.g. $s_A = 0.9$, $s_J = 0.5$, $f = 2$; then the eigenvalues are $(0.9 \pm 2.19)/2 = \{1.55, -0.65\}$.

Nonlinear multi-state models

Things get interesting now: e.g. epidemic model, $S(t+1) = S + (mN - mS - \beta SI)$, $I(t+1) = I + (\beta SI - mI - \gamma I)$.



Equilibria: straightforward algebra (solve $f(S^*, I^*) = \{S^*, I^*\}$). Stability: *linearization* of $f(S, I)$ near $\{S^*, I^*\}$.