Continuous-time deterministic models (ordinary differential equations) I: first-order models

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Advantages: conceptually simpler. No overshooting; e.g. chaotic dynamics impossible in < 3 dimensions. Easier to define sensible dynamics. Fewer difficulties with order of events. Often (?) more realistic.

Disadvantages: rigorous analysis is harder; solving the system requires calculus (integration). Numerical solutions harder.

(Side note on realism: is time "really" continuous or discrete? Hybrids . . .)

1 First-order models: analysis

Simple (univariate) equations (e.g. $dN/dt = rN$). (Compartmental models with a single compartment.)

Separable equations

These are equations that can be put in the form $dx/dt = Q(x)R(t)$, therefore we can separate them, integrate on both sides, and (hopefully) solve for $x(t)$.

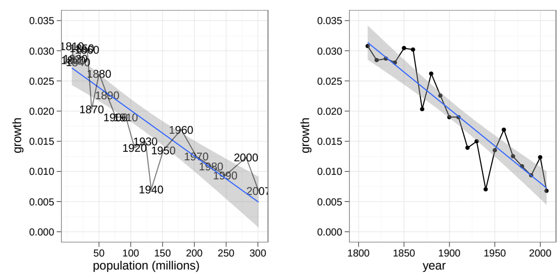
Examples: exponential, exponential plus constant (equivalent of affine models?), density-dependent mortality model.

Exponential equation: analog of geometric growth. $N(0)R^t = N(0)\exp(t \log R) = N(0)\exp(rt)$. Common in derivations of discrete-time rules, i.e. what do we get if we integrate some simple continuous process from t to $t + 1$?

Another standard trick: *partial fractions*. e.g. survival of a cohort under density-dependent mortality (see auxiliary derivation of *Beverton-Holt* equation) or logistic equation.

Logistic equation: important building block for any model (quadratic extension of exponential model). Solution: $K/(1 + C\exp(-rt))$ (can also rescale time, space to get *dimensionless* form $1/(1 + C\exp(-rt)))$. Properties of logistic growth: max. growth rate at $N = K/2$; inflection point in growth curve at $N = K/2$.

Population growth example (the book is silly!)



1.1 Bernoulli equations

Equations of the form

$$dx/dt + p(t)x = q(t)x^n$$

Can change variables to $w = x^{1-n}$, solve (possibly via integrating factors, see below).

von Bertalanffy growth function, theta-logistic equation.

1.2 Integrating factors

If a first-order linear ODE is in the form $dx/dt + P(t)x = Q(t)$ then if we define $\mu(t) = e^{\int P(t)\,dt}$ then $d(x\mu)/dt = \mu\,(dx/dt) + x\,d(\mu/dt)$; multiplying the original equation by μ on both sides makes everything work nicely.

2 First-order systems: graphical analysis

Graphical evaluation of 1-D systems: stability is trivial (none of this fancy stuff with the absolute value of the slopes). Lots we can automatically/qualitatively say about stability.

Finding the equilibria: possibly hard algebraically (as with discrete systems), but all we need to do is evaluate the derivative of the gradient function at the equilibrium and we're done.

Terminology: *(non)-autonomous.*

3 Numerical integration

Euler's method. *Don't use it!*

Runge-Kutta, etc etc etc.. "Stiff" systems.

In R: need to define gradient function.

```
> library(deSolve)
> ## parameters *must be in this order*
> ##    but names don't matter
> gradfun <- function(t,N,params) {
   with(c(as.list(N),as.list(params)),
        ## magic -- must return a list
        ## with gradient as the *first element*
        list(c(N=r*N*(1-N/K)),NULL))

 }
> desol <- lsoda(y=c(N=0.1),  ## init cond
                times=seq(0,10,by=0.1),
                func=gradfun,
                parms=c(r=1,K=5))
> ## handy for referring to columns
> desol <- as.data.frame(desol)
> with(desol,plot(time,N))
```