Continuous-time deterministic models (ordinary differential equations) I: derivations

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1 Beverton-Holt model

Density-dependent mortality of an initial cohort, over a fixed time T.

$$\begin{aligned}\frac{dN}{dt} &= -(a+bN) \\ \frac{dN}{(a+bN)N} &= -dt\end{aligned} \tag{1}$$

Separate by partial fractions:

$$\begin{aligned}\frac{1}{(a+bN)N} &= \frac{A}{a+bN} + \frac{B}{N} \\ AN + B(a+bN) &= 1 \\ aB + (A+bB)N &= 1 + 0 \cdot N\end{aligned} \tag{2}$$

so $B = 1/a$, $A = -bB = -b/a$.

$$\begin{aligned}\frac{dN}{(-b/aN)} + \frac{dN}{\frac{1}{a}(a+bN)} &= -dt \\ -\frac{a}{b}\log N + \frac{a}{b}\log(a+bN) &= -t + C'' \\ -\log N + \log(a+bN) &= -\frac{b}{a}t + C' \\ \frac{N}{a+bN} &= C\exp(-bt/a)\end{aligned} \tag{3}$$

Substituting $t = 0$, $C = N_0/(a+bN_0)$.

Now solve for N (ugh): call $C\exp(-bt/a) \equiv \phi$.

$$\begin{aligned}\phi &= \frac{N}{a+bN} \\ N &= \phi a + \phi bN \\ &= \frac{\phi a}{1-\phi b} \\ &= \frac{\frac{aN_0}{a+bN_0}e^{-bt/a}}{1 - \frac{bN_0}{a+bN_0}e^{-bt/a}} \\ &= \frac{N_0}{e^{bt/a} + (b/a)\left(e^{bt/a} - 1\right)N_0}\end{aligned} \tag{4}$$

2 Theta-logistic equation

$$\frac{dN}{dt} = rN\left(1 - \left(\frac{N}{K}\right)^{\theta}\right) \tag{5}$$

Use standard rescaling trick: set $\tau = rt$ (rescale time), $N' = NK^{\theta}$ (rescale space), hence

$$\frac{dN'}{d\tau} = N'(1 - N'^{\theta}) \tag{6}$$

Change of variables. Could crank through the formula, or use $N^{\theta} = 1/w$ (dropping the prime on N'). This gives us

$$\theta N^{\theta-1}\,dN = -w^{-2}\,dw,$$

$1/w \cdot 1/N\,dN = -w^{-2}\,dw$, or $dN = -N/w\,dw$.

Substitute in: $-N/w\,dw = N(1-\frac{1}{w})\,d\tau$ or

$$dw = w(1-w)\,d\tau \tag{7}$$

and we're back to the logistic equation!

Solve by partial fractions; reverse the w/N' substitution; reverse the N', τ substitution.

The answer should (??) be:

$$N(t) = (K^(-\theta) + (N(0)^(-\theta) - K^(-\theta))\exp(-r\theta t))^{-1/\theta} \tag{8}$$