Structured models

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Structured models. linear only. (Perhaps unrealistic but very useful for understanding short-term growth/growth when rare . . .)

e.g. Leslie matrix model: $\mathbf{A}$=0 except for first row (fecundities, ≥ 0) and first subdiagonal (survival, $0 < a_{i+1,i} \leq 1$). Lefkovitch model: the same except that more column entries can be non-zero (columns add to ≤ 1).

Time evolution of system: $\mathbf{x}(n) = \mathbf{A}^n\mathbf{x}(0)$. (Matrix multiplication in R: `%*%`; matrix powers, `matpow` in the `expm` package.

Irreducible non-negative matrix: For every ij there exists m such that $(\mathbf{A}^k)_{ij}$ is not zero. (There are no "dead-end" stages.)

Primitive matrix: There exists k such that for *all* ij, $(\mathbf{A}^k)_{ij}$ is positive. (There are loops of different lengths > 1.)

Perron-Frobenius theorem: for a non-negative, irreducible, primitive matrix, there is a single positive largest root; its associated eigenvector is the only positive (all-equal-sign) eigenvector.

Eigenvalues/eigenvectors: $\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$.

If the max. modulus of the eigenvalues $\max|\lambda|$ (also called *spectral radius*) > 1, then the population grows geometrically in the long-term; the associated eigenvalue is called the *stable age (stage) distribution.* (Transients may be different; e.g. start with no adults.) (Note eigenvectors can be scaled any way you like: $\sum e_i = 1$ is convenient for interpretation, R scales so that $\sum e_i^2 = 1$.)

Teasel model:

```
> options(digits=3)
> snames <- c("dseed1","dseed2",
              "smros","medros","lgros","flow")
> X <- matrix(c(0,0,0,0,0,322.38,
                0.966,0,0,0,0,0,
                0.013,0.010,0.125,0,0,3.448,
                0.007,0,0.125,0.238,0,30.17,
                0.008,0,0,0.245,0.167,0.862,
                0,0,0,0.023,0.75,0),
              byrow=TRUE,nrow=6,
              dimnames=list(snames,snames))
> library(expm)
> x200 <- (X%^%200) %*% c(1,0,0,0,0,0)
> as.vector(x200/sum(x200))

[1] 0.63690 0.26498 0.01217 0.06928 0.01208 0.00459

> e1 <- eigen(X) ## eigenvalues/vectors
> e1$values[1]   ## pop. growth rate

[1] 2.32+0i

> ## speed of approach to stable distribution:
> ##    ratio of first two eigenvalues
> Mod(e1$values[2]/e1$values[1])

[1] 0.762

> as.numeric(e1$vectors[,1]/sum(e1$vectors[,1]))

[1] 0.63690 0.26498 0.01217 0.06928 0.01208 0.00459

> sum(e1$vectors[,1]^2)

[1] 1+0i
```

`http://www.stanford.edu/~jhj1/teachingdocs/Jones-Leslie1-050208.pdf`

Markov models: conserved values. Columns sum to 1. Lead eigenvector now equal to 1; the lead eigenvector is the stable distribution.

If we have an absorbing state (matrix is not reducible!) then the final results might depend on starting conditions: which hole will we fall into?

Transition matrix is block-diagonal, with absorbing states forming an identity matrix (a non-absorbing, b absorbing states):

$$\begin{pmatrix} A_{b\times b} & 0_{b\times a} \\ B_{a\times b} & I_{a\times a} \end{pmatrix}$$

Can work out probability of ending up in a particular absorbing state, average length of time to get there.