PART V

Wavelets & Multiresolution Analysis

- ADDITIONAL REFERENCES:
  - www.wavelet.org
WAVELETS — OVERVIEW (I)

- What is wrong with FOURIER ANALYSIS ???
  - All spatial information is hidden in the PHASES of the expansion coefficients and therefore not readily available
  - Localized functions (“bumps”) tend to have a very complex representation in Fourier space
  - Local modification of the function affects its whole Fourier transform
  - If the dominant frequency changes in space, only average frequencies are encoded in Fourier coefficients

- Remedy — need an analysis tool that will encode both SPACE (TIME) and FREQUENCY information at the same time

- Following the convention, will work with TIME (t) and FREQUENCY (ω), rather than wavenumber (k)
WAVELETS — OVERVIEW (II)

• From Discrete Fourier Transform to Integral Fourier Transform — Consider the space \( L_2(\mathbb{R}) \) of square-integrable functions defined on \( \mathbb{R} \); if \( f \in L_2(\mathbb{R}) \) satisfies suitable decay conditions at \( \pm \infty \) (which??), the Discrete Fourier Transform can be replaced with the Integral Fourier Transform

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega t} \, dt
\]
\[
f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega
\]

• Interestingly, the Fourier Transforms (both discrete and integral) are constructed as “superpositions” of DILATIONS of the function \( w(x) = e^{ix} \) (\( w_k(t) = w(kt) \))

• Want to construct an integral transform using a basis function \( \psi \) which is very localized (a “wavelet”); we will therefore need:
  – DILATIONS
  – TRANSLATIONS
WAVELETS — GABOR TRANSFORM (I)

- The history begins with a **WINDOWED FOURIER TRANSFORM** known as the **GABOR TRANSFORM** (1946)

\[
(G_b^\alpha f)(\omega) = \int_{-\infty}^{\infty} \left(f(t)e^{-i\omega t}\right) g_\alpha(t-b) \, dt,
\]

where the **WINDOW FUNCTION** is given by \( g_\alpha(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4\alpha}} \) with \( \alpha > 0 \)

- Note that the Fourier transform of a Gaussian function is another Gaussian function, i.e.,

\[
\int_{-\infty}^{\infty} e^{-i\omega x} e^{ax^2} \, dx = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}
\]

- Note also that the window function has the following normalization

\[
\int_{-\infty}^{\infty} g_\alpha(t-b) \, db = \int_{-\infty}^{\infty} g_\alpha(x) \, dx = 1
\]

- Therefore, for the Gabor transform we obtain

\[
\int_{-\infty}^{\infty} (G_b^\alpha f)(\omega) \, db = \hat{f}(\omega), \quad \omega \in \mathbb{R}
\]

- Thus, the set \( \{G_b^\alpha f : b \in \mathbb{R}\} \) of Gabor transforms of \( f \) decomposes the Fourier transforms \( \hat{f} \) of \( f \) exactly to give its **LOCAL** spectral information
The **width** of the window function can be characterized by employing the notion of the **standard deviation**

\[
\Delta g_\alpha \triangleq \frac{1}{\|g_\alpha\|_2} \left\{ \int_{-\infty}^{\infty} x^2 g_\alpha^2(x) \, dx \right\}^{1/2}
\]

Note that for \( \alpha > 0 \) \( \Delta g_\alpha = \sqrt{\alpha} \)

Proof:
- \( \|g_\alpha\| = (8\pi\alpha)^{-1/4} \) can be evaluated setting \( \omega = 0 \) and \( a = (2\alpha)^{-1} \) in the expression for the Fourier transform of a Gaussian function
- \( \int_{-\infty}^{\infty} x^2 g_\alpha^2(x) \, dx \) can be evaluated differentiating twice the Fourier transform of a Gaussian function and again setting \( \omega = 0 \) and \( a = (2\alpha)^{-1} \)

Instead of localizing the Fourier transform of \( f \), the Gabor transform may equivalently be regarded as windowing \( f \) with the **window function** \( G_{b,\omega}^\alpha \)

\[
(G_{b}^\alpha f)(\omega) = (f, G_{b,\omega}^\alpha) = \int_{-\infty}^{\infty} f(t) \overline{G_{b,\omega}^\alpha(t)} \, dt, \quad G_{b,\omega}^\alpha(t) = \frac{e^{i\omega t}}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4\alpha}}
\]
Using the Parseval identity and noting that
\[ \hat{G}_{b,\omega}^\alpha(\eta) = e^{-ib(\eta-\omega)}e^{-\alpha(\eta-\omega)^2} \]
we obtain for the Gabor transform
\[
( G_b^\alpha f)(\omega) = (f, G_b^\alpha) = \frac{1}{2\pi} (\hat{f}, \hat{G}_{b,\omega}^\alpha) \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\eta)e^{ib(\eta-\omega)}e^{-\alpha(\eta-\omega)^2} d\eta \\
= e^{-ib\omega} \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} \left(e^{ib\eta}\hat{f}(\eta)\right) g_{1/4\alpha}(\eta - \omega) d\eta \\
= e^{-ib\omega} \frac{1}{2\sqrt{\pi\alpha}} (G_{\omega}^{1/4\alpha}\hat{f})(-b)
\]

- The third line (in red) indicates that up to a multiplicative factor \( \sqrt{\frac{\pi}{\alpha}} e^{-ib\omega} \)
  - the **windowed Fourier transform** of \( f \) with \( g_\alpha \) at \( t = b \),
  - the **windowed inverse Fourier transform** of \( \hat{f} \) with \( g_{1/4\alpha} \) at \( \eta = \omega \) are equal!
**WAVELETS — UNCERTAINTY PRINCIPLE (I)**

- Consider more general window functions \( w \in L_2(\mathbb{R}) \) which satisfy the requirement

\[
    tw(t) \in L_2(\mathbb{R})
\]

It can be shown that

- \( |t|^{1/2}w(t) \in L_2(\mathbb{R}) \)
- \( w \in L_1(\mathbb{R}) \)
- the Fourier transform \( \hat{w} \) is continuous
- \( \hat{w} \in L_2(\mathbb{R}) \)

Note, however, that in general \( x\hat{w}(x) \notin L_2(\mathbb{R}) \), therefore \( w \) may not in general be a **FREQUENCY WINDOW FUNCTION**

- If \( w \in L_2(\mathbb{R}) \) is chosen so that both \( w \) and \( \hat{w} \) satisfy the above condition, then the window Fourier transform

\[
    (\tilde{G}_b f)(\omega) = \int_{-\infty}^{\infty} \left( f(t)e^{-i\omega t} \right) \overline{w(t-b)} \, dt = (f, W_b, \omega),
\]

where \( W_b, \omega = e^{i\omega t}w(t-b) \), is called a **SHORT–TIME FOURIER TRANSFORM**
WAVELETS — UNCERTAINTY PRINCIPLE (II)

- We can define the **center** $x^*$ and **radius** $\Delta_w$ of $w$ as
  
  $$
x^* \equiv \frac{1}{\|w\|^2} \int_{-\infty}^{\infty} t |w(t)|^2 dt,
  \quad \Delta_w \equiv \frac{1}{\|w\|^2} \left\{ \int_{-\infty}^{\infty} (t - x^*)^2 |w(t)|^2 dt \right\}^{1/2}
  $$

- Then, $(\tilde{G}_{b,f})(\omega)$ gives local information on $f$ in the **TIME–WINDOW**
  
  $$[x^* + b - \Delta_w, x^* + b + \Delta_w]$$

- We can determine the **center** $\omega^*$ and the **radius** $\Delta_{\hat{\omega}}$ of the (frequency) window function $\hat{\omega}$ using formulae similar to the above

- Defining $V_{b,\omega}(\eta) \equiv \frac{1}{2\pi} \hat{W}_{b,\omega}(\eta) = \frac{1}{2\pi} e^{ib\omega} e^{-ib\eta} \hat{\omega}(\eta - \omega)$, which is also a window function with the center $\omega^* + \omega$ and radius $\Delta_{\hat{\omega}}$, we can write (using the Parseval identity)
  
  $$(\tilde{G}_{b,f})(\omega) = (f, W_{b,\omega}) = (\hat{f}, V_{b,\omega})$$

- Thus, $(\tilde{G}_{b,f})(\omega)$ also gives local spectral information about $t$ in the frequency window
  
  $$[\omega^* + \omega - \Delta_{\hat{\omega}}, \omega^* + \omega + \Delta_{\hat{\omega}}]$$
**Wavelets — Uncertainty Principle (III)**

- Therefore by choosing $w \in L^2(\mathbb{R})$, such that $xw(x) \in L^2(\mathbb{R})$ and $x\hat{w}(x) \in L^2(\mathbb{R})$, to define a windowed Fourier transform $(\tilde{G}_b f)(\omega)$ we obtain localization in a **TIME–FREQUENCY WINDOW**

$$[x^* + b - \Delta_w, x^* + b + \Delta_w] \times [\omega^* + \omega - \Delta_{\hat{w}}, \omega^* + \omega + \Delta_{\hat{w}}]$$

with area equal to $4\Delta_w \Delta_{\hat{w}}$

- In fact, there is a relation between possible time and frequency windows which is made precise in the following theorem

- **Heisenberg Uncertainty Principle** — Let $w \in L^2(\mathbb{R})$ be chosen so that $xw(x) \in L^2(\mathbb{R})$ and $x\hat{w}(x) \in L^2(\mathbb{R})$. Then

$$\Delta_w \Delta_{\hat{w}} \geq \frac{1}{2}$$

Furthermore, equality is attained if and only if

$$w(t) = ce^{i\alpha t} g_\alpha(t - b),$$

where $c \neq 0$, $\alpha > 0$, and $a, b \in \mathbb{R}$. 
**Wavelets — Uncertainty Principle (IV)**

- **Proof of the Heisenberg Uncertainty Principle**
  
  - Let us assume that the centers $x^*$ and $\omega^*$ are zero (if they are not, then we can modify $w$ as $\tilde{w}(t) = e^{-i\omega^*t}f(t + x^*)$)
  
  - We observe that

\[
\Delta_w^2 \Delta_{\tilde{w}}^2 = \frac{\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt \int_{-\infty}^{\infty} \omega^2 |\hat{w}(\omega)|^2 \, d\omega}{||w||_2^2 ||\hat{w}||_2^2}
= \frac{\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt \int_{-\infty}^{\infty} |w'(t)|^2 \, dt}{||w||_4^4}
\]

  - Using the Schwarz inequality we get

\[
\Delta_w^2 \Delta_{\tilde{w}}^2 \geq \frac{1}{||w||_2^2} \left[ \int_{-\infty}^{\infty} |t \bar{w}(t)w'(t)| \, dt \right]^2
\geq \frac{1}{||w||_4^4} \left[ \int_{-\infty}^{\infty} \frac{t}{2} [\overline{w}(t)w'(t) + \overline{w'}(t)w(t)] \, dt \right]^2
\geq \frac{1}{4||w||_4^4} \left[ \int_{-\infty}^{\infty} t(|w(t)|^2)' \, dt \right]^2
\]
Proof of the Heisenberg Uncertainty Principle — continued

Integrating by parts and noting that \( \lim_{|t| \to 0} \sqrt{t} f(t) = 0 \) (since \( |t|^{1/2} w(t) \in L_2(\mathbb{R}) \) seen earlier) we obtain

\[
\Delta_w^2 \Delta_{\hat{w}}^2 \geq \frac{1}{4\|w\|_2^4} \left[ \int_{-\infty}^{\infty} |w(t)|^2 \, dt \right]^2 = \frac{1}{4}
\]

An equality will be obtained when the Schwarz inequality becomes an equality; this implies that there exists \( b \in \mathbb{C} \) such that

\[
w'(t) = -2btw(t)
\]

so that there exists an \( a \in \mathbb{C} \) such that \( w(t) = ae^{-bt^2} \)

Thus the Gabor transform has the smallest possible time–frequency window.

The above Heisenberg Uncertainty Principle has far–reaching consequences.
INTEGRAL WAVELET TRANSFORM (I)

• The short–time Fourier transform has a RIGID time–frequency window, in the sense that its width ($\Delta \omega$) is unchanged for all frequencies analyzed; this turns out to be a limitation when studying functions with varying frequency content.

• The INTEGRAL WAVELET TRANSFORM provides a window which:
  – automatically narrows when focusing on high frequencies,
  – automatically widens when focusing on low frequencies.

• If $\psi \in L_2(\mathbb{R})$ satisfies the “admissibility” condition
  \[ C_\psi \triangleq \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty, \]
  then $\psi$ is called a BASIC WAVELET. Relative to every basic wavelet $\psi$, the INTEGRAL WAVELET TRANSFORM (IWT) in $L_2(\mathbb{R})$ is defined by
  \[ (W_\psi f)(a,b) \triangleq |a|^{\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \psi \left( \frac{x-b}{a} \right) dx, \quad f \in L_2(\mathbb{R}), \quad a \neq 0, b \in \mathbb{R}, \]
Hereafter we will assume that \( t\psi(t) \in L_2(\mathbb{R}) \) and \( \omega \hat{\psi}(\omega) \in L_2(\mathbb{R}) \), so that the basic wavelet \( \psi \) provides a time-frequency window with finite area.

From the above assumption it also follows that \( \hat{\psi} \) is a continuous function and therefore finiteness of \( C_\psi \) implies

\[
\hat{\psi}(0) = 0 \implies \int_{-\infty}^{\infty} \psi(t) \, dt = 0
\]

Setting

\[
\psi_{b;\alpha}(t) \triangleq |a|^{-\frac{1}{2}} \psi \left( \frac{t-b}{a} \right),
\]

the IWT can be written as \((W_\psi f)(b, a) = (f, \psi_{b;\alpha})\).

If the wavelet \( \psi \) has the center and radius given by \( t^* \) and \( \Delta_\psi \), respectively, then the function \( \psi_{b;\alpha} \) has its center at \( b + at^* \) and radius equal to \( a\Delta_\psi \).

Thus, the IWT provides local information about the function \( f \) in a time window

\[
[b + at^* - a\Delta_\psi, b + at^* + a\Delta_\psi]
\]

which narrows down as \( a \to 0 \).
**Integral Wavelet Transform (III)**

- Consider the Fourier transform of a basic wavelet

\[
\frac{1}{2\pi} \hat{\psi}_{b,a}(\omega) = \frac{|a|^{-\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \psi\left(\frac{t-b}{a}\right) \, dt = \frac{a|a|^{-\frac{1}{2}}}{2\pi} e^{-i\omega b} \hat{\psi}(\omega)
\]

- Suppose that \( \hat{\psi} \) has the center \( \omega^* \) and radius \( \Delta \hat{\psi} \). Defining \( \eta(\omega) \triangleq \hat{\psi}(\omega + \omega^*) \) we obtain a window function with center at the origin and unchanged radius.

- Applying the Parseval identity to the definition of the IWT we obtain

\[
(W_\psi f)(a, b) = \frac{a|a|^{-\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \eta(a\omega - \omega^*) \, d\omega,
\]

which, modulo multiplication by a constant factor and a linear frequency shift, localizes information about the function \( f \) to the FREQUENCY WINDOW

\[
\left[ \frac{\omega^*}{a} - \frac{1}{a} \Delta \psi, \frac{\omega^*}{a} + \frac{1}{a} \Delta \psi \right]
\]
INTEGRAL WAVELET TRANSFORM (IV)

- Note that the ratio of the **center frequency** $\omega^*/a$ to the **bandwidth** $2\Delta_\psi/a$

  $\frac{\text{center frequency}}{\text{bandwidth}} = \frac{\omega^*}{2\Delta_\psi}$

  is independent of the scaling $a$; thus, the bandwidth grows with frequency in an adaptive fashion (constant–Q filtering)

- Reconstruction of a function from its IWT

  Let $\psi$ be a basic wavelet, then $\forall f, g \in L_2(\mathbb{R})$

  $$\int_0^\infty \left[ \int_{-\infty}^\infty (W_\psi f)(b, a)\overline{(W_\psi g)(b, a)} \, db \right] \frac{da}{a^2} = \frac{1}{2} C_\psi(f, g)$$

  Furthermore, for any $f \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$ at which $f$ is continuous

  $$f(x) = \frac{2}{C_\psi} \int_0^\infty \left[ \int_{-\infty}^\infty (W_\psi f)(b, a)\psi_{b; a}(x) \, db \right] \frac{da}{a^2}$$

  Proof — using the Parseval identity, integrating with respect to $da/a^2$ and using the definition of $C_\psi$

  Note the role of the **admissibility** condition for $\psi$
**Discrete Wavelet Transform (I)**

- Consider the IWT at a discrete set of samples \( a = 2^{-j} \) and \( b = k2^{-j} \) for some \( j, k \in \mathbb{Z} \)

\[
(W_{\psi}f) \left( \frac{k}{2^j}, \frac{1}{2^j} \right) = \int_{-\infty}^{\infty} f(x) 2^{j/2} \psi(2^j x - k) \, dx = (f, \psi_{j,k})
\]

where

\[
\psi_{j,k} \triangleq 2^{j/2} \psi(2^j x - k)
\]

must be chosen so that \( \psi_{j,k} \) form a Riesz basis in \( L_2(\mathbb{R}) \) (i.e., the linear span of \( \psi_{j,k} \) with \( j, k \in \mathbb{Z} \) is dense in \( L_2(\mathbb{R}) \))

- If \( \psi_{j,k} \) with \( j, k \in \mathbb{Z} \) is a **Riesz Basis**, the the relation

\[
(\psi_{j,k}, \psi^{l,m}) = \delta_{j,l} \delta_{k,m}, \quad j, k, l, m \in \mathbb{Z}
\]

uniquely defines **another Riesz Basis** \( \psi^{l,m} \) known as the **Dual Basis**

- Thus, every function \( f \in L_2(\mathbb{R}) \) has a unique representation

\[
f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k}) \psi^{j,k}(x)
\]
**Discrete Wavelet Transform (II)**

- For the above representation to qualify as a *wavelet series*, the dual basis \( \psi_{j,k} \) must be obtained from some basic wavelet \( \tilde{\psi} \) by \( \psi_{j,k}(x) = \tilde{\psi}_{j,k}(x) \), where

\[
\tilde{\psi}_{j,k} \triangleq 2^{j/2} \tilde{\psi}(2^j x - k)
\]

- In general, \( \tilde{\psi} \) does not necessarily exist.

- If \( \psi \) is chosen so that \( \tilde{\psi} \) does exist, the pair \( (\psi, \tilde{\psi}) \) can be used interchangeably

\[
f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k}) \tilde{\psi}_{j,k}(x) = \sum_{j,k=-\infty}^{\infty} (f, \tilde{\psi}_{j,k}) \psi_{j,k}(x)
\]

- \( \psi \) and \( \tilde{\psi} \) are called *wavelet* and *dual wavelet*, respectively.

- If the basis \( \psi_{j,k} \) is orthogonal, i.e., \( \psi_{j,k} = \psi^{j,k} \) for \( j, k \in \mathbb{Z} \), we obtain an *orthogonal wavelet transform*

\[
f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k}) \psi_{j,k}(x)
\]
**Discrete Wavelet Transform (III)**

- Consider a wavelet $\psi$ and the Riesz basis $\psi_{j,k}$ it generates; for each $j \in \mathbb{Z}$, let $W_j$ denote the closure of the linear span of $\{\psi_{j,k} : k \in \mathbb{Z}\}$, i.e.,

$$W_j \triangleq \text{clos}_{L^2(\mathbb{R})}\{\psi_{j,k} : k \in \mathbb{Z}\}$$

- Evidently, $L^2(\mathbb{R})$ can be decomposed as a direct sum of the spaces $W_j$ (dots over pluses indicate “direct sums”)

$$L^2(\mathbb{R}) = \sum_{j \in \mathbb{Z}} W_j \triangleq \cdots + W_{-1} + W_0 + W_1 + \cdots$$

and therefore every function $f \in L^2(\mathbb{R})$ has a unique decomposition

$$f(x) = \cdots + g_1(x) + g_0(x) + g_1(x) + \cdots$$

where $g_j \in W_j$, $\forall j \in \mathbb{Z}$

- if $\psi$ is an orthogonal wavelet, then the subspaces $W_j \in L^2(\mathbb{R})$ are mutually orthogonal $W_j \perp W_l$, $j \neq l$ which means that

$$(g_j, g_l) = 0, \; j \neq l$$

where $g_j \in W_j$ and $g_l \in W_l$
**Discrete Wavelet Transform (IV)**

- Therefore, in such case, the direct sum becomes an **ORTHOGONAL SUM**
  \[ L_2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j \cong \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots \]

- Thus, an orthogonal wavelet \( \psi \) generates an **ORTHOGONAL DECOMPOSITION** of the space \( L_2(\mathbb{R}) \), as the functions \( g_j \) are
  - **UNIQUE**
  - **MUTUALLY ORTHOGONAL**
MULTIRESOLUTION ANALYSIS (I)

- For every wavelet $\psi$ (not necessarily orthogonal) we can consider the following space $V_j \in L_2(\mathbb{R})$, $\forall j \in \mathbb{Z}$

$$V_j = \cdots \hat{+} W_{j-2} \hat{+} W_{j-1}$$

- The subspaces $V_j$ have the following very interesting properties:
  1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$
  2. $\text{clos}_{L_2}(\bigcup_{j \in \mathbb{Z}} V_j) = L_2(\mathbb{R})$
  3. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
  4. $V_{j+1} = V_j \hat{+} W_j$, $j \in \mathbb{Z}$
  5. $f(x) \in V_j \iff f(2x) \in V_{j+1}$, $j \in \mathbb{Z}$

- Note that
  - In contrast to the subspaces $W_j$ which satisfy $W_j \cap W_l = \{0\}$, $j \neq l$, the sequence of subspaces $V_j$ is **NESTED** (1°)
  - Every $f \in L_2(\mathbb{R})$ can be approximated with **ARBITRARY ACCURACY** by its projections $P_j f$ on $V_j$ (2°)
MULTIRESOLUTION ANALYSIS (II)

- If the reference subspace $V_0$ is generated by a single scaling function $\phi \in L_2(\mathbb{R})$ in the sense that

$$V_0 = \text{clos}_{L_2(\mathbb{R})}\{\phi_{0,k} : k \in \mathbb{Z}\}$$

where

$$\phi_{j,k} \triangleq 2^{j/2}\phi(2^j x - k),$$

then all the subspaces $V_j$ are also generated by the same $\phi$ as

$$V_j = \text{clos}_{L_2(\mathbb{R})}\{\phi_{j,k} : k \in \mathbb{Z}\}$$

in the same way as the subspaces $W_j$ are generated by the wavelet $\psi$

- In the MULTIRESOLUTION ANALYSIS at a given scale $(j + 1)$
  - the subspace $V_j$ represents the “LARGE SCALE” features of the function
  - the subspaces $W_j$ represents the “SMALL SCALE” features (details) of the function
THE END