On an Attempt to Simplify the Quartapelle–Napolitano Approach to Computation of Hydrodynamic Forces in Open Flows

Bartosz Protas

Department of Mathematics & Statistics, McMaster University, Hamilton, ON, Canada

Abstract

In this paper we are interested in the Quartapelle–Napolitano approach to calculation of forces in viscous incompressible flows in exterior domains. We study the possibility of deriving a simpler formulation of this approach which might lead to a more convenient expression for the hydrodynamic force, but conclude that such a simplification is, within the family of approaches considered, impossible. This shows that the original Quartapelle–Napolitano formula is in fact "optimal" within this class of approaches.

Key words: Navier–Stokes equation, Hydrodynamic Forces, Variational Methods
PACS: 47.10.ad, 47.27.Ak, 47.11.-j

1 Introduction

Calculation of hydrodynamic forces acting on an object immersed in a fluid is one of the central objectives in many applied problems in Fluid Dynamics. In this investigation we analyze the possibility of extending the approach to calculation of forces proposed by Quartapelle and Napolitano (1983). We will be concerned with incompressible flows in unbounded exterior domains (Figure 1a). In some derivations we will also consider truncations Ω_1 of the domain Ω obtained by imposing an exterior boundary Γ_1 (Figure 1b). We will fix the origin of the coordinate system at the obstacle and will assume that the obstacle remains motionless with the fluid velocity vanishing on its boundary. We will also assume that there is a uniform flow $U_\infty \mathbf{e}_1$ at infinity ($\mathbf{e}_1$ is the unit vector corresponding to the OX axis). The fluid motion is governed by the Navier–Stokes system representing conservation of mass

Email address: bprotas@mcmaster.ca (Bartosz Protas).
URL: http://www.math.mcmaster.ca/bprotas (Bartosz Protas).

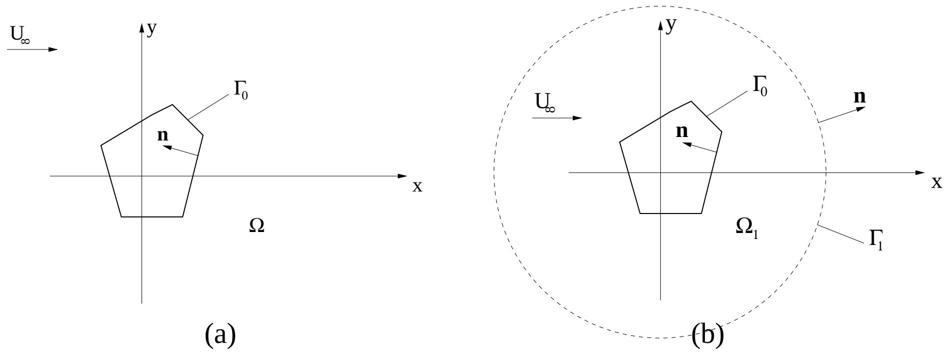

Fig. 1. Schematic of the flow past an obstacle Γ_0 in (a) an unbounded exterior domain Ω and (b) an exterior domain Ω_1 with an outer boundary Γ_1.

and momentum. This system of equations will be assumed to have the following form

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u}\times\boldsymbol{\omega} + \boldsymbol{\nabla}\frac{\mathbf{u}^2}{2} + \boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}\times\boldsymbol{\omega} = 0 \qquad \text{in } \Omega\times[0,T] \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad \text{in } \Omega\times[0,T] \tag{1b}$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0 \qquad \text{in } \Omega \tag{1c}$$

$$\mathbf{u}\big|_{\Gamma_0} = 0 \qquad \text{in } [0,T] \tag{1d}$$

$$\mathbf{u} \longrightarrow U_\infty \mathbf{e}_1 \qquad \text{in } [0,T] \text{ for } |\mathbf{x}|\to\infty, \tag{1e}$$

where: $\mathbf{u} = [u_1, u_2, u_3]$ is the velocity field, $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ is the vorticity, p is the pressure, ν represents the coefficient of the kinematic viscosity (the density of the fluid is assumed equal to unity), $\mathbf{u}_0$ is the initial condition, T represents the end of the time interval considered and $\mathbf{x} = [x_1, x_2, x_3]$ is the position vector. Given an object with a boundary Γ_0 characterized by the local unit normal vector $\mathbf{n}$ facing into the object (Figure 1a,b), the hydrodynamic force acting on this object is, by definition, given by the following expression

$$\mathbf{F} = \mathbf{F}^p + \mathbf{F}^\nu = \oint_{\Gamma_0} p\mathbf{n}\,d\sigma - \nu\oint_{\Gamma_0}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right]\mathbf{n}\,d\sigma = \oint_{\Gamma_0} p\mathbf{n}\,d\sigma + \nu\oint_{\Gamma_0}\mathbf{n}\times\boldsymbol{\omega}\,d\sigma. \tag{2}$$

The velocity gradient is defined as $[\boldsymbol{\nabla}\mathbf{u}]_{ij} = \frac{\partial u_i}{\partial x_j}$ and the two forms of the viscous term $\mathbf{F}^\nu$ are equivalent due to the identity $\oint_{\Gamma_0}(\boldsymbol{\nabla}\mathbf{u})^T\mathbf{n}\,d\sigma = 0$ valid for all incompressible fields $\mathbf{u}$. The arguments that we will elaborate in this paper will be valid in both 2D and 3D domains; for the sake of simplicity of exposition, however, the main proof will be restricted to the 2D case with its generalization to 3D being quite straightforward.

It is often convenient to solve equations of fluid motion (1) in one of the so–called “non–primitive” formulations involving only vorticity and velocity, or streamfunction [see, e.g. Gresho (1991); Quartapelle (1993)]. In such cases one does not have direct access to the pressure required to evaluate $\mathbf{F}^p$. Similar situation arises also in experimental investigations where the Particle Image Velocimetry (PIV) measurements are capable of extracting instantaneous velocity and vorticity fields with systematically increasing resolution in space and time [see, e.g. Rockwell (2000)]. Unavailability of pressure in such approaches motivates the need for alternative ways of calculating the hydrodynamic force in which pressure is not needed. In the literature several methods have been proposed, all relying on suitable manipulation of the Navier–Stokes system (1). Below we will briefly review the most important results; derivation of some of these approaches will be analyzed in detail in the following Section. We also remark that, in view of the assumptions made, these expressions will not include terms corresponding to the motion of the obstacle. This lack of generality, however, does not affect the main point of the paper.

The best known approach, popularized by Saffman (1992), expresses the force in terms of the vorticity impulse as

$$\mathbf{F} = -\frac{1}{D-1}\frac{d}{dt}\int_{\Omega}\mathbf{x}\times\boldsymbol{\omega}\,d\Omega, \tag{3}$$

where $D = 2,3$, is the spatial dimension. While providing an interesting insight into the relationship between the force and vorticity dynamics, this approach has the disadvantage that integration is extended over the whole infinite domain. Consequently, vorticity at very large distances from the obstacle must be included which can be quite difficult in both numerical simulations and PIV measurements. In addition, the time derivative present in (3) tends to amplify noise. As an alternative, Noca, Shiels, and Jeon (1997, 1999) proposed a family of formulas with the generic form

$$\mathbf{F} = -\frac{1}{D-1}\frac{d}{dt}\int_{\Omega_1}\mathbf{x}\times\boldsymbol{\omega}\,d\Omega + \Big[\text{integral over } \Gamma_1\Big] + \Big[\text{integral over } \Gamma_0\Big], \tag{4}$$

where integration is restricted to the truncated domain Ω_1 and the far field contribution is contained in the integral over Γ_1. These formulas no longer require integration over an infinite domain, but still suffer from the presence of the time derivative. Furthermore, evaluation of the fluxes involved in the integrals over Γ_1 may be complicated.

A different approach was proposed by Quartapelle and Napolitano (1983) where, before integrating over the domain, the momentum equation (1a) is multiplied by the gradient $\boldsymbol{\nabla}\eta_a$ of a harmonic function η_a which satisfies a Neumann–type boundary condition $\mathbf{n}\cdot\boldsymbol{\nabla}\eta_a = -\mathbf{n}\cdot\mathbf{a}$ on Γ_0 and whose gradient decays to zero at the outer boundary. As a result, the hydrodynamic force in the direction of the vector $\mathbf{a}\in\mathbb{R}^D$ is given by the expression

$$F_a = \mathbf{F}\cdot\mathbf{a} = -\int_{\Omega}\boldsymbol{\nabla}\eta_a\cdot(\mathbf{u}\times\boldsymbol{\omega})\,d\Omega + \nu\oint_{\Gamma_0}(\boldsymbol{\nabla}\eta_a+\mathbf{a})\cdot(\mathbf{n}\times\boldsymbol{\omega})\,d\sigma. \tag{5}$$

We remark that in the above expression the two terms involving the function η_a represent the contributions from the pressure force $\mathbf{F}^p\cdot\mathbf{a}$. Formula (5) has the advantage that, apart from the absence of the time–derivative, the integrand expression in the area integral includes a factor that rapidly decays with the distance from the obstacle. As a result, formula (5) is much more convenient to apply in numerical simulations where resolution of the velocity and vorticity fields is usually decreased far from the obstacle, and/or PIV measurements where data is usually confined to a finite domain. This method has been further developed by Chang (1992); Howe (1995); Chang and Lei (1996), Chang, Su, and Lei (1998) which included a generalization for the compressible case, Protas, Styczek, and Nowakowski (2000) and Pan and Chew (2002). This approach has also been the method of choice for force calculations in several investigations employing the Vortex Methods [see, e.g. Smith and Stansby (1988); Chang and Chern (1991); Stansby and Slaouti (1993); Clarke and Tutty (1994); Protas, Styczek, and Nowakowski (2000); Cheng, Liu, and Lam (2001)] and appears as a promising possibility for calculating force based on data obtained from PIV measurements (Wesfreid, private communication). A similar approach has been used by Wells for theoretical investigations (Wells, 1996, 1998). It has been recognized, however, that the approach leading to formula (5) has certain shortcomings. We note that the expression for the pressure force $\mathbf{F}^p$ involves a boundary integral term proportional to the viscosity ν. In order to evaluate this term and the term representing viscous stresses, the distribution of vorticity on the boundary must be available which in many applications is rather inconvenient (in grid–based numerical methods and in PIV this may require construction of complicated differentiation stencils, whereas in vortex methods complex interpolation schemes may be needed). The purpose of the present paper is to examine the possibility of simplifying the Quartapelle–Napolitano approach in a way to alleviate these difficulties, i.e., express the force with a formula akin to (5), but without boundary terms involving data other than the boundary conditions for problem (1). We will attempt this by replacing $\boldsymbol{\nabla}\eta_a$ in the derivation of (5) with a more general function. It will be proven, however, that such a simplification is not, in fact, possible. Therefore, the Quartapelle–Napolitano formula (5) can be regarded as "optimal" within this family of variational approaches (the term "optimal" is not used here in its strictly mathematical sense, but rather implies that formula (5) represents an approach more convenient than other from the computational point of view).

The structure of the paper is as follows: in Section 2 we revisit the derivation of formula (5) in a more general setting and suggest formally the new approach, in Section 3 we state and prove a theorem showing that the desired simplification is not in fact possible; some consequences of the presented arguments and conclusions are discussed in Section 4.

2 The Variational Formulation — A General Approach

In this Section we analyze the derivation of the Quartapelle–Napolitano formula (5) with the purpose of modifying this derivation in such a way that the boundary terms involving vorticity would no longer be present. We will investigate this possibility as a generalization of the standard approach introduced by Quartapelle and Napolitano (1983). First we show that the different approaches to calculation of forces mentioned in Section 1 and the approach we are about to investigate can in fact be derived using the following general procedure:

Procedure 1 *(1) choose a function $\boldsymbol{\gamma} \in [H^1(\Omega)]^D$, where $[H^1(\Omega)]^D$ denotes the Sobolev space of vector–valued functions with square–integrable derivatives in Ω, such that*

$$\mathcal{B}\boldsymbol{\gamma}\big|_{\Gamma_0} = \mathcal{B}\mathbf{a}, \tag{6}$$

where $\mathcal{B} : \boldsymbol{\gamma}\big|_{\Gamma_0} \to \mathcal{B}\boldsymbol{\gamma}\big|_{\Gamma_0}$ is a linear operator acting on the boundary values (traces) of the function $\boldsymbol{\gamma}$,

(2) multiply the momentum equation (1a) *by $\boldsymbol{\gamma}$ and integrate over the truncated domain Ω_1*

$$\int_{\Omega_1} \boldsymbol{\gamma} \cdot \left[\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} + \boldsymbol{\nabla} \frac{\mathbf{u}^2}{2} \right] d\Omega = \int_{\Omega_1} \boldsymbol{\gamma} \cdot \left[-\boldsymbol{\nabla} p - \nu \boldsymbol{\nabla} \times \boldsymbol{\omega} \right] d\Omega, \tag{7}$$

(3) use integration by parts and relation (6) *valid on the boundary to extract from* (7) *the terms corresponding to* (2)*,*

(4) assume that $\Gamma_1 \to \infty$ which, given the assumptions on the behavior of $\boldsymbol{\gamma}$ and $\mathbf{u}$ for large $|\mathbf{x}|$ will remove the integrals defined on Γ_1.

More specifically, step 4 of Procedure 1 requires an assumption that the decay of the field $(\mathbf{u} - U_\infty \mathbf{e}_1)$ for $|\mathbf{x}| \to \infty$ be sufficiently rapid [$O(|\mathbf{x}|^{-2})$ in 2D and $O(|\mathbf{x}|^{-3})$ in 3D]. We remark that *steady* solutions of the Navier–Stokes system do not necessarily satisfy these assumptions [see, e.g. Galdi (1994)]. As a result, some of following expressions may, somewhat paradoxically, be inapplicable in the case of steady flows.

In order to obtain a unique function $\boldsymbol{\gamma}$, condition (6) has to be supplemented with an additional condition defined in the domain Ω. We will illustrate now how the above general procedure can lead, for different choices of this additional condition, and hence the function $\boldsymbol{\gamma}$, to formulas (3), (4) and (5). Then, a new approach will be formulated by requiring of the function $\boldsymbol{\gamma}$ to be characterized by still more general properties. In all cases we will calculate the projection of the force on an arbitrary vector $\mathbf{a}$ (by choosing $\mathbf{a} = \mathbf{e}_i$, $i = 1, \dots, D$, where $\mathbf{e}_i$ is the unit vector associated with the i–th axis of the coordinate system, we will obtain components of the force in the corresponding directions).

The impulse formula (3) is obtained trivially by choosing

$$\boldsymbol{\gamma} = \mathbf{a} \text{ in } \Omega, \text{ hence, by extension, } \mathcal{B} = \mathrm{Id} \;\Rightarrow\; \boldsymbol{\gamma}\big|_{\Gamma_0} = \mathbf{a}, \tag{8}$$

i.e., the function $\boldsymbol{\gamma}$ is constant and given by the vector $\mathbf{a}$ everywhere. Following Procedure 1 and using standard vector identities [see, e.g. Noca, Shiels, and Jeon (1999)] we obtain

$$\mathbf{F}\cdot\mathbf{a} = -\frac{\mathbf{a}}{D-1}\cdot\frac{d}{dt}\int_{\Omega}\mathbf{x}\times\boldsymbol{\omega}\,d\Omega \tag{9}$$

which is in fact equivalent to formula (3) dotted with the vector $\mathbf{a}$. By abandoning step 4 of the procedure, i.e., retaining a truncated domain Ω_1, we would obtain an expression for $\mathbf{F}\cdot\mathbf{a}$ equivalent to dotting (4) with the vector $\mathbf{a}$.

The Quartapelle–Napolitano formula (5) is obtained by choosing the function $\boldsymbol{\gamma}$ in the form $\boldsymbol{\gamma} = -\boldsymbol{\nabla}\eta_a$, where η_a satisfies the following Neumann problem for the Laplace equation

$$\begin{cases} \boldsymbol{\nabla}\cdot\boldsymbol{\gamma} = -\Delta\eta_a = 0 & \text{in } \Omega, \\ \mathcal{B} = (\mathbf{n},\cdot) & \Rightarrow\; (\mathbf{n},\boldsymbol{\gamma}\big|_{\Gamma_0}) = -\mathbf{n}\cdot\boldsymbol{\nabla}\eta_a\big|_{\Gamma_0} = \mathbf{n}\cdot\mathbf{a}, \\ \boldsymbol{\gamma}\to 0 & \text{for } |\mathbf{x}|\to\infty. \end{cases} \tag{10}$$

where $(\cdot,\cdot)$ represents the standard Euclidean inner product. Following the steps outlined in Procedure 1 and employing transformations described in detail in Protas, Styczek, and Nowakowski (2000), we can express the pressure force as

$$\mathbf{F}^p\cdot\mathbf{a} = -\int_{\Omega}\boldsymbol{\nabla}\eta_a\cdot(\mathbf{u}\times\boldsymbol{\omega})\,d\Omega + \nu\oint_{\Gamma_0}\boldsymbol{\nabla}\eta_a\cdot(\mathbf{n}\times\boldsymbol{\omega})\,d\sigma. \tag{11}$$

The second term on the right–hand side in (11) is similar, but not equal, to the term representing the viscous stresses in (2). In order to obtain an expression for the total force, the viscous term $\mathbf{F}^{\nu}\cdot\mathbf{a}$ must be added to (11) which will result in formula (5). The form of second term on the right–hand side in (5), which can be rewritten as

$$\nu\oint_{\Gamma_0}(\boldsymbol{\nabla}\eta_a + \mathbf{a})\cdot(\mathbf{n}\times\boldsymbol{\omega})\,d\sigma = -\nu\oint_{\Gamma_0}\boldsymbol{\omega}\cdot[\mathbf{n}\times(\boldsymbol{\nabla}\eta_a+\mathbf{a})]\,d\Omega, \tag{12}$$

may suggest that we could redefine the function $\boldsymbol{\gamma}$ by adjusting the boundary condition (6) in such a way as to get rid of this term altogether. This is the motivation for a more general approach that we consider formally below.

In this approach we will thus employ a function $\boldsymbol{\gamma}$ which should satisfy the following set of conditions

$$\begin{cases} \boldsymbol{\nabla}\times\boldsymbol{\gamma} = 0 & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{\gamma} = 0 & \text{in } \Omega, \\ \mathcal{B} = Id \;\Rightarrow \boldsymbol{\gamma} = \mathbf{a} & \text{on } \Gamma_0, \\ \boldsymbol{\gamma}\to 0 & \text{for } |\mathbf{x}|\to\infty. \end{cases} \tag{13}$$

We notice that this problem in fact represents an augmented version of the original problem (10) in which the constraint of a vanishing curl is added and the boundary conditions are specified for all components of $\boldsymbol{\gamma}$, rather than just the wall–normal component. The hope is that these additional assumptions would make it possible to derive an expression for the hydrodynamic force simpler than the original formula (5). The important issue of consistency of system (13) will be specifically addressed in Section 3. In principle, solution to problem (13) can be constructed using the Helmholtz–Hodge decomposition (Quartapelle, 1993)

$$\boldsymbol{\gamma} = -\boldsymbol{\nabla}\varphi - \boldsymbol{\nabla}\times\boldsymbol{\psi}, \tag{14}$$

where the two functions φ and $\boldsymbol{\psi}$, corresponding to the potential and the solenoidal part, can be found by solving the system of equations

$$\begin{cases} -\Delta\varphi = 0 & \text{in } \Omega, \\ -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi} = 0 & \text{in } \Omega, \\ \quad \boldsymbol{\nabla}\cdot\boldsymbol{\psi} = 0 & \text{in } \Omega, \\ \quad \mathbf{n}\cdot\boldsymbol{\nabla}\varphi + \mathbf{n}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\psi}) = -\mathbf{n}\cdot\mathbf{a} & \text{on } \Gamma_0, \\ \quad \mathbf{n}\times\boldsymbol{\nabla}\varphi + \mathbf{n}\times\boldsymbol{\nabla}\times\boldsymbol{\psi} = -\mathbf{n}\times\mathbf{a} & \text{on } \Gamma_0, \\ \quad \boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\boldsymbol{\psi} \to 0 & \text{for } |\mathbf{x}| \to \infty, \end{cases} \tag{15}$$

where the third equation is a "gauge" condition added to close the system. We remark that system (15) is equivalent to (13)–(14). As is well known (Quartapelle, 1993; Richardson and Cornish, 1977), in general decomposition (14) is not unique, but there are several ways to make it unique by prescribing appropriate boundary conditions for the potentials φ and $\boldsymbol{\psi}$. However, for our purposes here it is sufficient to leave the boundary conditions in the coupled form present in (15). In the remaining part of this Section we will *formally* use solutions of system (15) together with Procedure 1 to derive an apparently very appealing expression for the hydrodynamic force.

Since the derivations to follow are new, we will present them in some detail. We begin by integrating by parts the terms on the right–hand side of (7)

$$\begin{aligned} -\int_{\Omega_1} \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}p\,d\Omega &= \int_{\Omega_1} p\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}d\Omega - \oint_{\Gamma_0\bigcup\Gamma_1} p\mathbf{n}\cdot\boldsymbol{\gamma}d\sigma \\ &= -\oint_{\Gamma_0} p\mathbf{n}\cdot\mathbf{a}\,d\sigma - \oint_{\Gamma_1} p\mathbf{n}\cdot\boldsymbol{\gamma}d\sigma, \\ -\nu\int_{\Omega_1} \boldsymbol{\gamma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\omega})\,d\Omega &= -\nu\int_{\Omega_1} \boldsymbol{\omega}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\gamma})\,d\Omega - \nu\oint_{\Gamma_0\bigcup\Gamma_1} \boldsymbol{\gamma}\cdot(\mathbf{n}\times\boldsymbol{\omega})\,d\sigma \\ &= -\nu\oint_{\Gamma_0} \mathbf{a}\cdot(\mathbf{n}\times\boldsymbol{\omega})\,d\sigma - \nu\oint_{\Gamma_1} \boldsymbol{\gamma}\cdot(\mathbf{n}\times\boldsymbol{\omega})\,d\sigma. \end{aligned}$$

When $\Gamma_1 \to \infty$, the assumptions concerning the asymptotic behavior of $\boldsymbol{\gamma}$ and $\mathbf{u}$ for large $|\mathbf{x}|$ imply that the integrals on Γ_1 vanish. Thus, we obtain for the terms on the right–hand side in (7)

$$\int_{\Omega} \boldsymbol{\gamma} \cdot [-\nabla p - \nu \nabla \times \boldsymbol{\omega}]\, d\Omega = -\mathbf{a} \cdot \oint_{\Gamma_0} p\mathbf{n}\, d\sigma - \nu \mathbf{a} \cdot \oint_{\Gamma_0} \mathbf{n} \times \boldsymbol{\omega}\, d\sigma = -\mathbf{F} \cdot \mathbf{a}. \qquad (16)$$

We now proceed to analyze the terms on the left–hand side in (7) and begin with the time–derivative term in which we express the velocity field in terms of the stream vector $\boldsymbol{\Psi}$ as $\mathbf{u} = \nabla \times \boldsymbol{\Psi}$

$$\begin{aligned} \int_{\Omega_1} \boldsymbol{\gamma} \cdot \frac{\partial \mathbf{u}}{\partial t}\, d\Omega &= \int_{\Omega_1} \boldsymbol{\gamma} \cdot (\nabla \times \frac{\partial \boldsymbol{\Psi}}{\partial t})\, d\Omega = \int_{\Omega_1} \frac{\partial \boldsymbol{\Psi}}{\partial t} \cdot (\nabla \times \boldsymbol{\gamma})\, d\sigma + \oint_{\Gamma_0 \bigcup \Gamma_1} \boldsymbol{\gamma} \cdot (\mathbf{n} \times \frac{\partial \boldsymbol{\Psi}}{\partial t})\, d\sigma \\ &= \oint_{\Gamma_0} \mathbf{a} \cdot (\mathbf{n} \times \frac{\partial \boldsymbol{\Psi}}{\partial t})\, d\sigma + \oint_{\Gamma_1} \boldsymbol{\gamma} \cdot (\mathbf{n} \times \frac{\partial \boldsymbol{\Psi}}{\partial t})\, d\sigma \end{aligned}$$

Letting again $\Gamma_1 \to \infty$, by (1e) we have $\boldsymbol{\Psi} \to \boldsymbol{\Psi}_\infty \triangleq [0, -\frac{1}{2}U_\infty x_3, \frac{1}{2}U_\infty x_2]$ which is time–independent, so that the integral on Γ_1 vanishes and we obtain

$$\int_{\Omega} \boldsymbol{\gamma} \cdot \frac{\partial \mathbf{u}}{\partial t}\, d\sigma = \oint_{\Gamma_0} (\mathbf{n} \times \mathbf{a}) \cdot \frac{\partial \boldsymbol{\Psi}}{\partial t}\, d\sigma. \qquad (17)$$

The streamvector $\boldsymbol{\Psi}$ is defined via the system of equations

$$\begin{cases} \Delta \boldsymbol{\Psi} = -\boldsymbol{\omega} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\Psi} = 0 & \text{in } \Omega, \\ \mathbf{n} \cdot (\nabla \times \boldsymbol{\Psi}) = 0 & \text{on } \Gamma_0, \\ \boldsymbol{\Psi} \to \boldsymbol{\Psi}_\infty & \text{for } |\mathbf{x}| \to \infty. \end{cases} \qquad (18)$$

In order to evaluate (17) we need the boundary value $\boldsymbol{\Psi}\big|_{\Gamma_0}$ of the stream vector. It is known [see, e.g. Quartapelle (1993)] that the boundary value $\boldsymbol{\Psi}\big|_{\Gamma_0}$ can be expressed, up to a time–dependent constant, entirely in terms of the boundary data for velocity (this construction is illustrated for the simpler 2D case in Appendix).

We keep the form of the second term on the left–hand side in (7) unchanged, whereas as regards the third term we proceed using integration by parts as follows

$$\begin{aligned} \int_{\Omega_1} \boldsymbol{\gamma} \cdot \nabla \left(\frac{\mathbf{u}^2}{2} \right) d\Omega &= -\int_{\Omega_1} (\nabla \cdot \boldsymbol{\gamma}) \frac{\mathbf{u}^2}{2}\, d\Omega + \oint_{\Gamma_0 \bigcup \Gamma_1} \mathbf{n} \cdot \boldsymbol{\gamma} \frac{\mathbf{u}^2}{2}\, d\sigma \\ &= \oint_{\Gamma_0} \mathbf{n} \cdot \mathbf{a} \frac{\mathbf{u}^2}{2}\, d\sigma + \oint_{\Gamma_1} \mathbf{n} \cdot \boldsymbol{\gamma} \frac{\mathbf{u}^2}{2}\, d\sigma. \end{aligned}$$

As before, when $\Gamma_1 \to \infty$, the integral over Γ_1 vanishes and we obtain

$$\int_{\Omega} \boldsymbol{\gamma} \cdot \nabla \left(\frac{\mathbf{u}^2}{2} \right) d\sigma = \oint_{\Gamma_0} \mathbf{n} \cdot \mathbf{a} \frac{\mathbf{u}^2}{2}\, d\sigma. \qquad (19)$$

Thus, putting together (7), (16), (17) and (19), we obtain what appears to be a new expression for the hydrodynamic force in which the boundary integrals are expressed *entirely* in terms of the boundary conditions for the problem (1)

$$\mathbf{F}\cdot\mathbf{a} = \int_\Omega \boldsymbol{\gamma}\cdot(\mathbf{u}\times\boldsymbol{\omega})\,d\Omega - \oint_{\Gamma_0}(\mathbf{n}\times\mathbf{a})\cdot\frac{\partial\boldsymbol{\Psi}}{\partial t}\,d\sigma + \oint_{\Gamma_0}\mathbf{n}\cdot\mathbf{a}\frac{\mathbf{u}^2}{2}\,d\sigma.$$

Assuming, as we do in this investigation, that the boundary velocity vanishes and invoking arguments presented in Appendix reduces this expression to a particularly simple form

$$\mathbf{F}\cdot\mathbf{a} = \int_\Omega \boldsymbol{\gamma}\cdot(\mathbf{u}\times\boldsymbol{\omega})\,d\Omega, \tag{20}$$

where the function $\boldsymbol{\gamma}$ should satisfy (13). We emphasize that, in contrast to (5), expression (20) does not contain any boundary integrals involving vorticity which, as argued in Section 1, are awkward to evaluate in numerical simulations and using data from PIV measurements. In Section 3 we will prove that, regrettably, a function $\boldsymbol{\gamma}$ required to derive (20) cannot in fact be constructed, because system (13) does not admit any solutions.

3 Proof of Non–Existence of the Function γ

In this Section we present a simple proof that, if $\mathbf{a}$ is *constant* vector, system (13) does not in fact admit any solutions. It is a well–known fact of vector analysis (Girault and Raviart, 1979) that, given the divergence and curl of a vector field in a bounded domain, this vector field can be reconstructed so that it will satisfy only *one* scalar boundary condition. As a matter of fact, this vector field may satisfy boundary conditions on other components as well, but only when the divergence and curl are "special", in the sense that they satisfy additional constraints, e.g., they come from solutions of the Navier–Stokes equation. Thus, in general, problems (13)–(15) are overdetermined. Below we show that for the choice of the boundary data in (13) (a constant vector) required for the derivation of (20), the solution does not indeed exist. For the sake of simplicity of the proof, we restrict our attention here to the 2D case:

Theorem 1 *Given a constant vector* $\mathbf{a}\in\mathbb{R}^2$*, system* (13) *has no solutions in 2D.*

PROOF. Assume for the moment that a function $\boldsymbol{\gamma}$ satisfying (13) exists. Since the 2D case is considered, we have $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, 0]$ and $\frac{\partial}{\partial x_3} \equiv 0$. The first two equations in (13) now reduce to

$$\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} = \frac{\partial\gamma_1}{\partial x_1} + \frac{\partial\gamma_2}{\partial x_2} = 0, \tag{21a}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\gamma} = \mathbf{e}_3\left(\frac{\partial\gamma_2}{\partial x_1} - \frac{\partial\gamma_1}{\partial x_2}\right) = \mathbf{0}. \tag{21b}$$

We define now $z \triangleq x_1 + ix_2$ as a point in the complex plane $\mathbb{C}$, where $i = \sqrt{-1}$ is the imaginary unit. Thus, relations (21) represent the Cauchy–Riemann conditions for two conjugate functions γ_1 and γ_2 which can therefore be regarded as the real and imaginary part of an analytic function $W(z) \triangleq \gamma_1 + i\gamma_2$. In an unbounded exterior domain Ω this analytic function can be represented as a Laurent series

$$W(z) = \sum_{k=-\infty}^{\infty} c_k z^k,$$

with the expansion coefficients $c_k \in \mathbb{C}$, $k = \ldots, -1, 0, 1, \ldots$. Without loss of generality, we assume that the contour Γ_0 is a unit circle (the exterior of any other sufficiently regular contour can be transformed into the exterior of a unit circle using a suitable conformal mapping). The boundary conditions for the problem (13) are equivalent to

$$W(e^{i\theta}) = \cdots + c_{-2}e^{-2i\theta} + c_{-1}e^{-i\theta} + c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \cdots = a, \tag{22a}$$

$$W(z) \to 0 \quad \text{for } z \to \infty, \tag{22b}$$

where $a = a_1 + ia_2$. We observe that, because of (22b), $c_k = 0$ for $k \geq 0$. Then, however, there is no choice of the remaining expansion coefficients c_k, $k < 0$, that can make the series (22a) equal to a constant for all values of θ. Thus, an analytic function satisfying conditions (22) does not exist, from which we conclude that system (13) in 2D does not admit any solutions. □

The proof in the 3D case can be constructed using analogous methods of the potential theory.

4 Conclusions

In this paper we showed how a family of well known approaches to calculation of forces in viscous incompressible flows in exterior domains, including the Quartapelle–Napolitano formula, can be derived in a generic way by making different choices of the vector field $\boldsymbol{\gamma}$ on which the momentum equation is projected. We then considered a potentially appealing simplification of the Quartapelle–Napolitano approach in which the terms involving the boundary vorticity are absent. It was obtained formally by requiring that the function $\boldsymbol{\gamma}$ have more general properties than used in the original approach. It was, however, proved that a function with such properties cannot be constructed, hence indicating that the original Quartapelle–Napolitano formula is "optimal" within this family of approaches.

Acknowledgments

The author wishes to thank the anonymous referees for their insightful comments. This work was carried out when the author was a visiting scholar at École Supérieure de Physique et de Chimie Industrielles (Paris) whose generous financial support is gratefully acknowledged.

Appendix Boundary Value of the Streamfunction

In 2D flows velocity can be expressed in terms of streamfunction ψ as $\mathbf{u} = [u, v] = \mathbf{e}_3 \times \boldsymbol{\nabla}\psi$, where $\mathbf{e}_3$ is the unit vector associated with the OZ axis. The boundary value of the streamfunction can be determined, for instance, from the boundary condition $\frac{\partial\psi}{\partial s}\big|_{\Gamma_0} = \mathbf{u}\cdot\mathbf{n}$, where s is the arc–length coordinate along the boundary. Hence

$$\psi(t,s) = \int_{s_0}^{s} u_n(t,s')\,ds' + C(t), \tag{23}$$

where s_0 is some arbitrarily chosen arc–length coordinate and $u_n \triangleq \mathbf{u}\cdot\mathbf{n}$. As shown in Girault and Raviart (1979), while the constant $C(t)$ may be determined requiring the pressure to be single–valued in the domain Ω, it will in general remain a function of time. Hence, in the 2D case, integral (17) can be expressed for a stationary contour Γ_0 as

$$\begin{aligned}\oint_{\Gamma_0} (n_x a_y - n_y a_x)\frac{\partial\psi}{\partial t}\,d\sigma &= \oint_{\Gamma_0} (n_x a_y - n_y a_x)\frac{\partial}{\partial t}\left[\int_{s_0}^{s} u_n(t,s')\,ds' + C(t)\right] d\sigma \\ &= \oint_{\Gamma_0} (n_x a_y - n_y a_x)\frac{\partial}{\partial t}\left[\int_{s_0}^{s} u_n(t,s')\,ds'\right] d\sigma \\ &\quad + \dot{C}(t)\oint_{\Gamma_0} (n_x a_y - n_y a_x)\,d\sigma \\ &= \oint_{\Gamma_0} (n_x a_y - n_y a_x)\frac{\partial}{\partial t}\left[\int_{s_0}^{s} u_n(t,s')\,ds'\right] d\sigma,\end{aligned} \tag{24}$$

where we observed that $\oint_{\Gamma_0} (n_x a_y - n_y a_x)\,d\sigma = a_y \oint_{\Gamma_0} n_x\,d\sigma - a_x \oint_{\Gamma_0} n_y\,d\sigma = 0$. Hence, expression (24) may be nonvanishing only if the boundary conditions for problem (1) are time–dependent.

References

Chang, C.-C., 1992. Potential Flow and Forces for Incompressible Viscous Flow. Proc. R. Soc. Lond. A **437**, 517–525.

Chang, C. C., Chern, R. L., 1991. Vortex Shedding from an Impulsively Started Rotating and Translating Circular Cylinder. J. Fluid Mech. **233**, 265–298.

Chang, C.-C., Lei, S.-Y., 1996. On the Sources of Aerodynamic Forces: Steady Flow around a Cylinder or a Sphere. Proc. R. Soc. Lond. A **452**, 2369–2395.

Chang, C.-C., Su, J.-Y., Lei, S.-Y., 1998. On Aerodynamic Forces for Viscous Compressible Flow. Theoret. Comput. Fluid Dyn. **10** 71–90.

Cheng M., Liu. G. R., Lam, K. Y., 2001. Numerical simulation of flow past a rotationally oscillating cylinder. Comp. and Fluids **30**, 365–392.

Clarke N. R., Tutty, O. R., 1994. Construction and Validation of a Discrete Vortex Method for the 2–Dimensional Incompressible Navier–Stokes Equations. Comp. and Fluids **23**, 751–783.

Galdi, G., 1994. An Introduction to the Mathematical Theory of Navier–Stokes Equations, Vol. I: Linearized Steady Problems, Vol. II: Nonlinear Steady Problems. Springer.

Girault, V., Raviart, P. -A., 1979. Finite element approximation of the Navier-Stokes equations Springer.

Gresho, P. .M., 1991. Incompressible Fluid Dynamics: Some Fundamental Formulation Issues. Ann. Rev. Fluid Mech. **23**, 413–453.

Howe, M. S., 1995. On the Force and Moment on a Body in an Incompressible Fluid, with Application to Rigid Bodies and Bubbles at High and Low Reynolds Numbers. Q. J. Mech. Appl. Math. **48**, 401–426.

Noca, F., Shiels, D., Jeon, D., 1997. Measuring Instantaneous Fluid Dynamic Forces on Bodies, using only Velocity Fields and their Derivatives. J. Fluids and Struct. **11**, 345–350.

Noca, F., Shiels, D., Jeon, D., 1999. A comparison of methods for evaluating time-dependent fluid dynamic forces on bodies, using only velocity fields and their derivatives. J. Fluids and Struct. **13**, 551–578.

Pan, L. S., Chew, Y. T., 2002. A General Formula for Calculating Forces on a 2-D Arbitrary Body in Incompressible Flow. J. Fluids Struc. **16**, 71–82.

Protas, B., Styczek, A., Nowakowski, A., 2000. An Effective Approach to Computation of Forces in Viscous Incompressible Flows. J. Comp. Phys. **159**, 231–245.

Quartapelle, L., 1993. Numerical Solution of the Incompressible Navier-Stokes Equations, Birkhäuser.

Quartapelle, L., Napolitano, M., 1983. Force and Moment in Incompressible Flows. AIAA Journal, Vol. **21**, 911–913.

Richardson, R. C. Cornish, A. R. H., 1977. Solution of three–dimensional incompressible flows, J. Fluid Mech. **82**, 309–319.

Rockwell, D., 2000. Imaging of unsteady separated flows: global interpretation with particle image velocimetry. Exp. in Fluids [Suppl.], S255–S273.

Saffman, P. E., 1992. Vortex Dynamics. Cambridge University Press.

Smith, P. A., Stansby, P. K., 1988. Impulsively Started Flow Around a Circular

Cylinder by the Vortex Method. J. Fluid Mech. **194**, 45–77.
Stansby P. K., Slaouti, A., 1993. Simulation of Vortex Shedding Including Blockage by the Random Vortex and Other Methods. Int. J. Num. Meth. Fluids **17**, 1003–1013.
Wells, J. C., 1996. A geometrical interpretation of force on a translating body in rotational flow. Phys. Fluids **8**, 442–450.
Wells, J. C., 1998. Calculating the Impulse and Angular Impulse of Image Vorticity by Integrals over Free Vorticity. Proc. R. Soc. Lond. A **454**, 2791–2809.