Model theory of tracial von Neumann algebras

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Introduction/History

- Traditionally, applications of model theory have come through the use of first order logic.
- Difficulties arise when one wishes to capture the underlying topology.
- The discrete ultraproduct plays a key foundational role via the theorem of Łoś.
- Almost in parallel ultraproducts were being used in places where the underlying structure was a pointed metric space - Banach spaces, von Neumann algebras, C*-algebras, asymptotic cones.
- Until recently there was no logical counterpart for this use of ultraproducts (early attempts by Keisler, Henson).

Continuous model theory - an example

- We wish to consider a tracial von Neumann algebra M as a logical structure.
- The relevant functions are $+, \times, *, 0, 1$ and multiplication by scalars from C thought of as unary functions.
- The trace will be thought of as a relation and we have a metric arising from the 2-norm $\|x\|_2 = \sqrt{tr(xx^*)}$ (here we use a normalized trace). We will almost never mention it but to make the general theory go smoothly, relations are assumed to be real-valued and so trace is really two relations, tr^r and tr^i for the real and imaginary part of the trace.
- We also have the operator norm; it plays a subtle role when considering M as a logical structure.

Continuous model theory - the operator norm

Suppose that $r \geq 0$ and B_r is the ball of operator norm $\leq r$. Then

- B_r is complete with respect to the 2-norm.
- For any of our functions, if we restrict the inputs to B_r then that function is uniformly continuous with respect to the 2-norm and there is a uniform bound on the operator norm of the output.
- Trace is uniformly continuous and bounded when restricted to B_r.

Continuous model theory

- A language for continuous model theory consists of a special symbol d intended as a metric, function symbols and relation symbols. Function and relation symbols come endowed with bounds and uniform continuity moduli as with the example of tracial von Neumann algebras.
- A metric structure is an interpretation of the symbols of the language. In particular, a metric structure is a metric space (X, d) where the metric symbol is interpreted as the metric d together with a scale $\sigma : X \to [0, \infty)$ such that for every $r \geq 0$:
  - $\sigma^{-1}([0, r])$ is complete with respect to d;
  - function symbols are interpreted on X so that they are uniformly continuous wrt d and bounded on $\sigma^{-1}([0, r])$ as specified by the language;
  - relation symbols are interpreted on X so that they are also uniformly continuous wrt d and bounded on $\sigma^{-1}([0, r])$ as specified by the language.

Ultraproducts

- Fix metric structures M_i for $i \in I$ for some language L and an ultrafilter U on I.
- The ultraproduct $\prod_{i \in I} M_i / U$ is defined via a pre-structure M' which has underlying set

$$\{\bar{m} : \text{for some } B, \{i : \sigma_i(m_i) \leq B\} \in U\}$$

- The scale σ on the this set is defined by

$$\sigma(\bar{m}) = \lim_{i \to U} \sigma(m_i)$$

- Relations are defined similarly (which includes the metric): for a relation R and $\bar{m}^1, \ldots, \bar{m}^n$ from M'

$$R(\bar{m}^1, \ldots, \bar{m}^n) = \lim_{i \to U} R^{M_i}(\bar{m}^1_i, \ldots, \bar{m}^n_i)$$

- Relations are bounded and uniformly continuous with respect to d because each of the component relations were required to be by the language.

Ultraproducts, cont'd

- Functions are defined coordinatewise; the demands of uniform continuity and boundedness again follow from the specification of the language.
- The relation corresponding to the metric is now only a pseudo-metric on M' and the ultraproduct is obtained by quotienting.
- For tracial von Neumann algebras, this construction is the usual ultraproduct construction. If all of the M_i's are the same metric structure M, we call this an ultrapower and we write M^U.

Syntax

- Terms are formed by composing function symbols and variables as in first order logic.
- If R is an n-ary relation symbol and $\tau_1, \ldots, \tau_n$ are terms then $R(\tau_1, \ldots, \tau_n)$ is a formula - sometimes called basic formulas.
- If $f : R^n \to R$ is a continuous function and $\varphi_1, \ldots, \varphi_n$ are formulas then $f(\varphi_1, \ldots, \varphi_n)$ is a formula.
- If $r \geq 0$ and φ is a formula then both $\sup_{\sigma(x) \leq r} \varphi$ and $\inf_{\sigma(x) \leq r} \varphi$ are formulas.
- Sentences are formulas with no free variables.

Syntax, cont'd

Terms and formulas are interpreted in metric structures as you would expect. The key points are:

- Every term is interpreted as a function on the metric structure which is both bounded and uniformly continuous when the scale is restricted.
- Although we are allowing arbitary continuous functions on the reals as connectives, inductively, all formulas have a bounded range and are uniformly continuous when the scale is restricted.
- Because of the previous comment, the interpretation of sup and inf formulas is well-defined.
- Sentences take on real values in a given metric structure.
- If M is a metric structure then the theory of M, $Th(M)$ is the function which assigns to every sentence its value in M or equivalently the set of sentences which evaluate to 0.

Syntax and axioms for tracial von Neumann algebras

- Terms in the language for tracial von Neumann algebras are just *-polynomials in many variables.
- The only relation is trace and so the only basic formulas look like $tr(p(\bar{x}))$ for some *-polynomial p.
- Certainly one can write down sentences that express most of the basic properties of tracial von Neumann algebras:

Axioms

- The axioms for a C-algebra with a compatible involution *; for example, for all $r \geq 0$,

$$\sup_{\|x\| \leq r} d((xy)^*, y^* x^*)$$

- Axioms expressing properties of the trace; for example,

$$tr(x + y) = tr(x) + tr(y)$$

- Call this theory T_{tr}.

We call a theory universal if all of its axioms are of the form $\sup \varphi$ where φ is quantifier-free. Notice that T_{tr} is universal.

Theorem

T_{tr} axiomatizes the class of tracial von Neumann algebras.

Formulas in ultraproducts

Theorem (Łoś Theorem)

Suppose we have metric structures M_i for $i \in I$ for some language L and an ultrafilter U on I. Fix a formula $\varphi(\bar{x})$ in L then if M is $\prod_{i \in I} M_i / U$ we have

$$\varphi^M(\bar{m}) = \lim_{i \to U} \varphi^{M_i}(\bar{m}_i)$$

In particular, if φ is a sentence then

$$\varphi^M = \lim_{i \to U} \varphi^{M_i}$$

Type spaces

- Suppose M is a metric structure and $B \subseteq M$. If $a \in M^U$ for some U then the type p of a over A is the function from all formulas over B to R such that for a formula φ and parameters $b \in B$,

$$\varphi(x, b) \mapsto \varphi(a, b)$$

- a is said to realize p, ($a \models p$).
- The set of all types over B whose scale is $\leq r$ is denoted $S^r(B)$. If one restricts oneself only to instances of a single formula $\varphi(x, y)$ then the set of functions is denoted $S^r_\varphi(B)$.

Metrics on type spaces

- There is a metric on $S^r(B)$ defined by

$$d(p,q) = \inf\{d(a,b) : a \models p, b \models q \text{ in some } M^U\}$$

- We also define a metric on $S^r_\varphi(B)$ by

$$d_\varphi(p,q) = sup_{b \in B}|\varphi(p,b) - \varphi(q,b)|$$

Stability and the order property

Definition

A theory T is stable if for all separable models M of T, all formulas $\varphi(x, y)$ and all $r \geq 0$, $S^r_\varphi(M)$ is separable.

Definition

M has the order property if there is a formula $\varphi(x, y)$ and $r < s$ such that for every N, there are $c_i \in M$ for $i \leq N$, of bounded scale such that $\varphi(c_i, c_j) = r$ if $i \leq j$ and $\varphi(c_i, c_j) = s$ if $i > j$.

Stability, cont'd

Lemma

If M is a II_1 factor then M has the order property.

Proof.

Let $x = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$ and let $y = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}$. Then we have $\|x\|_2 = 1 = \|y\|_2$. Also $[x, y] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ and $\|[x, y]\|_2 = 2$. For $1 \leq i \leq n - 1$ let

$$a_i = \bigotimes_{j=0}^{i} x \otimes \bigotimes_{j=i+1}^{n-1} I \text{ and } b_i = \bigotimes_{j=0}^{i} I \otimes y \otimes \bigotimes_{j=i+2}^{n-1} I$$

So in M_{2^n}, $\varphi(x_1, y_1, x_2, y_2) = \|[x_1, y_2]\|_2$ orders the $a_i b_i$'s. □

Stability, cont'd

Theorem

The following are equivalent for separable metric structures M

- *$Th(M)$ is stable.*
- *M does not have the order property.*
- *All countable non-principal ultrapowers of M are necessarily isomorphic.*

Theorem

For separable tracial von Neumann algebras M, $Th(M)$ is stable iff M is type I.

Corollary

($\neg$ CH) If M is a II_1 factor then M has non-isomorphic ultrapowers.

Question of McDuff

- If M is a separable II_1 factor and U, V are non-principal ultrafilters on $\mathbb{N}$, McDuff asked if $M' \cap M^U$ is necessarily isomorphic to $M' \cap M^V$.
- The following are equivalent for a separable II_1 factor M:
  - For all nonprincipal ultrafilters U and V on $\mathbb{N}$ the relative commutants $M' \cap M^U$ and $M' \cap M^V$ are necessarily isomorphic.
  - Some (all) relative commutant(s) of M is (are) type I (in fact are abelian).
  - Some (all) relative commutant(s) of M are stable.

Some questions

- What are the complete continuous first order theories of II_1 factors?
- What are the universal theories of II_1 factors?
- This is equivalent to the CEP!
- General fact from model theory that if M and N are metric structures then $Th_\forall(M) = Th_\forall(N)$ iff $Th_\exists(M) = Th_\exists(N)$.
- As a corollary, CEP is equivalent to the microstate conjecture.
- Even without CEP, $Th_\forall(R)$ is the maximal universal theory of II_1 factors. General model theory guarantees that there is a minimal universal theory of II_1 factors
- If S is any separable model of the minimal universal theory then it follows that every separable II_1 factor embeds into an ultrapower of S (poor man's CEP).