The model theory of $\mathcal{R}$

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May 30, 2013

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Outline

- How do you know if you are doing model theory?
- Some basics of continuous model theory for von Neumann algebras
- Proof of concept: some preliminary results
- Quantifier complexity
- Decidability

When is it model theory?

- Model theory focuses on a logical study of a class of structures.
- The emphasis here is “the class" over the “logical".
- How can you tell if a class might be amenable to a model theoretic study?
- Ultraproducts!
- The key observation is usually identifying a class of structures closed under ultraproducts.

Ultraproducts

- The discrete or set-theoretic ultraproduct has had many applications in applied model theory; number theoretic examples like the Ax-Kochen theorem from diophantine geometry would be typical.
- Banach spaces via the positive bounded logic of Ward Henson.
- Geometric group theory - asymptotic cones: Gromov, Van den Dries, Wilkie and now studied by M. Luther.
- C*-algebras via normed ultraproducts
- The study of II_1 factors via tracial ultraproducts going back to Sakai (and Wright) and McDuff.
- I want to consider II_1 factors as a case study.

Tracial ultraproducts

- Suppose that A is a von Neumann algebra. We say that A is tracial if it has a faithful, normal trace τ i.e. a positive linear functional τ which is both faithful ($\tau(a^*a) = 0$ implies $a = 0$), normal ($\tau(1) = 1$) and satisfies $\tau(xy) = \tau(yx)$. We write $\|a\|_2$ for the usual 2-norm induced by such a trace.
- If A_i for $i \in I$ are tracial von Neumann algebras with traces τ_i and U is an ultrafilter on I, one forms the tracial ultraproduct as follows:
- Let

  $$\ell^\infty(\prod_{i\in I} A_i) = \{\bar{a} \in \prod_{i\in I} A_i : \text{for some } M, \|a_i\| \leq M \text{ for all } i \in I\}$$

  and

  $$c_U = \{\bar{a} \in \ell^\infty(\prod_{i\in I} A_i) : \lim_{i\to U} \|a_i\|_2 = 0\}$$

- The ultraproduct is then $\prod_{i\in I} A_i/U := \ell^\infty(\prod_{i\in I} A_i)/c_U$.

The only metric structures we need

- Fix a tracial von Neumann algebra i.e. a von Neumann algebra A with a faithful, normal trace τ.
- Consider the structure where the underlying metric space is the operator norm unit ball $A_1 = \{x \in A : \|x\| \leq 1\}$ with the metric given by the 2-norm, $d(x, y) = \|x - y\|_2$.
- The functions we highlight (those “in the language") are all *-polynomials which map the operator norm unit ball of *any* tracial von Neumann algebra back into itself. For instance, $\frac{x+y}{2}, xy, x^*, \lambda x$ for $|\lambda| \leq 1$ etc.
- We also highlight the real and imaginary parts of the fixed trace.
- One key element of the general theory of metric structures is that all of these functions and relations are uniformly continuous with respect to the 2-norm.

The logic of metric structures

- Basic formulas will be of the form $Re(\tau(p(\bar{x})))$ and $Im(\tau(p(\bar{x})))$ where p is one of the *-polynomials we fixed.
- Quantifier-free formulas will be of the form $f(\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}))$ where $f : R^k \to R$ is a continuous function and $\varphi_1, \ldots, \varphi_k$ are basic formulas.
- Arbitrary formulas are obtained by "quantifying" over the variables using either sup or inf over the operator norm unit ball.
- So an arbitrary formula has the form:

$$Q^1_{x_1 \in B_1} Q^2_{x_2 \in B_1} \ldots Q^k_{x_k \in B_1} \varphi(x_1, \ldots, x_n)$$

  where each Q^i is either sup or inf and φ is a quantifier-free formula.

Interpreting formulas

- If we consider any formula $\varphi(\bar{x})$ and substitute elements $\bar{a}$ from a tracial von Neumann algebra A then $\varphi^A(\bar{a})$, φ evaluated at $\bar{a}$, is a number.
- If a formula has no free variables we call it a sentence and when we evaluate it in an algebra, a sentence is assigned a number.
- The theory of an algebra A in continuous logic is the function from sentences φ to numbers φ^A which assigns their value in A; we write $Th(A)$ for this function.
- It is equivalent to determine the set of sentences in a given algebra which evaluate to 0. In fact, we can determine $Th(A)$ from knowing the zero set on positive sentences.

Elementary classes

We say that a class of structures K is elementary if there is a set of sentences T such that $A \in K$ iff $\varphi^A = 0$ for all $\varphi \in T$.

Theorem (FHS)

1. *The class of all tracial von Neumann algebras is elementary.*
2. *The class of all II_1 factors is elementary.*

Elementary maps

If $A \subseteq B$ are two algebras then we say this embedding is elementary if for all formulas $\varphi(\bar{x})$ and $\bar{a} \in A$, $\varphi^A(\bar{a}) = \varphi^B(\bar{a})$.

Theorem (Łoś Theorem)

Suppose A_i are tracial von Neumann algebras for all $i \in I$, U is an ultrafilter on I, $\varphi(\bar{x})$ is a formula and $\bar{a} \in \prod_{i \in I} A_i/U$ then

$$\varphi(\bar{a}) = \lim_{i \to U} \varphi^{A_i}(\bar{a}_i)$$

It follows that the diagonal embedding of A into A^U is always elementary; in particular, $Th(A) = Th(A^U)$.

Property Γ

- If A is a II_1 factor and U is a non-principal ultrafilter on $\mathbb{N}$, a relative commutant of A in A^U, written $A' \cap A^U$ is

$$\{B \in A^U : B \text{ commutes with all } C \in A\}$$

- We say that A has property Γ if $A' \cap A^U \neq \mathbb{C}$.
- Having property Γ is independent of the choice of ultrafilter. In fact, property Γ is an elementary property.
- Indeed the sentences φ_n, for all $n \in N$, express property Γ where φ_n is

$$\sup_{x_1,\dots,x_n \in B_1} \inf_{y \in B_1} \left(\sum_{i=1}^{n} \|[x_i, y]\|_2 + |\tau(y)| + \|1 - y^* y\|_2\right)$$

Property Γ, cont'd

$\mathcal{R}$ has property Γ. In fact, McDuff showed that for a separable II_1 factor A either

- A does not have property Γ,
- A has property Γ, $A' \cap A^U$ is abelian and determined up to isomorphism by A, or
- A has property Γ and $A' \cap A^U$ has type II_1. This property became known as "being McDuff". She asked if the isomorphism type here was unique.

Theorem (FHS)

A tracial von Neumann algebra is stable iff it has type 1.

Corollary ($\neg$ CH)

If A is McDuff then it has non-isomorphic relative commutants.

Quantifier complexity

- Any two embeddings of $\mathcal{R}$ into $\mathcal{R}^\omega$ are unitarily equivalent.
- The diagonal embedding of $\mathcal{R}$ into $\mathcal{R}^\omega$ is elementary so any embedding of $\mathcal{R}$ into any model of $Th(\mathcal{R})$ is elementary ($\mathcal{R}$ is a prime model of its theory).
- One common reason model theoretically for this behaviour is that the given theory has quantifier elimination i.e. for any formula $\varphi(\bar{x})$ and $\epsilon > 0$ there is a quantifier-free formula $\psi(\bar{x})$ such that

$$\sup_{\bar{x} \in B_1} |\varphi(\bar{x}) - \psi(\bar{x})| \leq \epsilon$$

  is part of the theory.
- So, does $Th(\mathcal{R})$ have quantifier elimination?

Quantifier complexity, cont'd

- No! A paper of Nate Brown's contains the following calculation: If $\Gamma = SL_3(Z) * Z$ then $L(\Gamma)$ is $\mathcal{R}^\omega$-embeddable; in fact it has an automorphism α and embedding $\pi : L(\Gamma) \to \mathcal{R}^\omega$ for which α is not implemented by a unitary.
- But $L(\Gamma) \rtimes_\alpha Z$ is also $\mathcal{R}^\omega$-embeddable and this rules out quantifier elimination.
- In fact, with a little more work we show that the theory of tracial von Neumann algebras does not have a model companion - it had been conjectured that $Th(\mathcal{R})$ was such.
- Last straw: maybe $Th(\mathcal{R})$ is model complete - this would show that every formula is approximated by sup formulas.

Theorem (Goldbring, H., Sinclair)

If $Th(\mathcal{R})$ is model complete then CEP fails!

Connes' embedding problem

- Does every separable II_1 factor embed into $\mathcal{R}^\omega$?
- Formulas which have only sup (inf) quantifiers are called universal (existential). We write $Th_\forall(A)$ ($Th_\exists(A)$) for the universal (existential) theory of A. We can determine these by just looking at positive sentences which evaluate to 0.
- General fact: If $A \subseteq B$ then $Th_\forall(B) \subseteq Th_\forall(A)$.
- $\mathcal{R} \hookrightarrow A$ for any II_1 factor so $Th_\forall(A) \subseteq Th_\forall(\mathcal{R})$.
- $\mathcal{R} \prec \mathcal{R}^\omega$ so $Th_\forall(\mathcal{R}) = Th_\forall(\mathcal{R}^\omega)$. It follows then that CEP holds iff $Th_\forall(A) = Th_\forall(\mathcal{R})$ for all II_1 factors A.

Microstate conjecture

- Fact: $Th_\forall(A) = Th_\forall(B)$ iff $Th_\exists(A) = Th_\exists(B)$.
- It is immediate that CEP holds iff the microstate conjecture is true i.e. For any II_1 factor A, $\epsilon > 0$, *-polynomials $p_1(\bar{x}), \ldots, p_n(\bar{x})$ and $\bar{a} \in A$ there is $\bar{b} \in \mathcal{R}$ (alternatively, there is N and $\bar{b} \in M_N$) such that for all $i = 1, \ldots, n$,

$$|tr(p_i(\bar{a})) - tr(p_i(\bar{b}))| \leq \epsilon$$

Even without CEP

- $Th_\forall(\mathcal{R})$ is maximal among universal theories of II_1 factors; it follows by Łoś' theorem that there is a minimal universal theory i.e. there is a separable II_1 factor $\mathcal{S}$ such that for all II_1 factors A, $Th_\forall(\mathcal{S}) \subseteq Th_\forall(A)$.
- Again, it is immediate that for any separable II_1 factor A, $A \hookrightarrow \mathcal{S}^\omega$ (a poor man's resolution to CEP).
- Note: $Th_\forall(\mathcal{S}) = Th_\forall(\mathcal{R})$ iff CEP holds.
- Good question: what could $\mathcal{S}$ look like?

CEP and decidability

- The theory of II_1 factors has a recursively enumerable set of axioms i.e. it is possible to give an algorithm to list a set of continuous sentences, the models of which are exactly the class of II_1 factors.
- If CEP holds then there is an algorithm that would take a dense set of universal sentences and for every such φ and every $\epsilon > 0$ would return a value which would be the value of φ in any II_1 factor to within ϵ.
- If CEP holds the same would be true if universal was replaced by existential in the previous statement and with some additional work, one can even get an algorithm that approximates values for $\exists\forall$-sentences.
- CEP is equivalent to the decidability of the universal theories of all type II_1 tracial von Neumann algebras.