Model theory of operator algebras: the next generation

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Outline

- Continuous model theory and metric structures
- Model theory and the CEP
- Quantifier elimination and model completeness
- Stability and ultrapowers
- Classes of operator algebras and omitting types
- On beyond continuous model theory: von Neumann algebras

Continuous model theory

- Until further notice, the only structures we will talk about are tracial von Neumann algebras or C*-algebras.
- Basic formulas in these two cases will be of the form $Re(tr(p(\bar{x})))$ and $\|p(\bar{x})\|$ respectively where p is a *-polynomial with complex coefficients in several variables.
- Quantifier-free formulas will be of the form $f(\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x}))$ where $f : R^k \to R$ is a continuous function and $\varphi_1, \ldots, \varphi_k$ are basic formulas.
- Arbitrary formulas are obtained by "quantifying" over the variables using either sup or inf over an operator norm ball of radius N.
- So a formula has the form wlog:

$$Q^1_{x_1 \in B_{N_1}} Q^2_{x_2 \in B_{N_2}} \ldots Q^k_{x_k \in B_{N_k}} \varphi(x_1, \ldots, x_n)$$

where each Q^i is either sup or inf and φ is a quantifier-free formula.

Continuous model theory, cont'd

- Notice that if we consider any formula $\varphi(\bar{x})$ and substitute elements from some algebra A then the value of $\varphi(\bar{a})$ is some number.
- If a formula has no free variables we call it a sentence and when we evaluate it in an algebra, a sentence is assigned a number.
- The theory of an algebra A in continuous logic is the function from sentences φ to numbers φ^A which assigns their value in A; we write $Th(A)$ for this function.
- It is equivalent to determine the set of sentences in a given algebra which evaluate to 0. In fact, we can determine $Th(A)$ from knowing the zero set on positive sentences.
- Formulas which have only sup (inf) quantifiers are called universal (existential). We write $Th_\forall(A)$ ($Th_\exists(A)$) for the universal (existential) theory of A. Again, we can determine these by just looking at positive sentences.

Continuous model theory, cont'd

- There are two uses of the adjective elementary which will be relevant: if $A \subseteq B$ are two algebras then we say this embedding is elementary if for all formulas $\varphi(\bar{x})$ and $\bar{a} \in A$, $\varphi^A(\bar{a}) = \varphi^B(\bar{a})$.

Theorem (Łoś Theorem)

Suppose A_i are algebras for all $i \in I$, U is an ultrafilter on I, $\varphi(\bar{x})$ is a formula and $\bar{a} \in \prod_{i \in I} A_i/U$ then

$$\varphi(\bar{a}) = \lim_{i \to U} \varphi^{A_i}(\bar{a}_i)$$

- It follows that the diagonal embedding of A into A^U is always elementary; in particular, $Th(A) = Th(A^U)$.

Continuous model theory, cont'd

- The other use of elementary is: we say that a property of a class of algebras is elementary if it can be expressed by a set of sentences - it can be axiomatized.
- The class of tracial von Neumann algebras is elementary; in fact, it can be axiomatized by universal sentences.
- The class of II_1 factors is elementary.
- Maybe more interestingly, property Gamma and being McDuff are both elementary properties.
- The class of C*-algebras is elementary; in fact, it is universally axiomatized as well.
- Properties like being $\mathcal{Z}$-stable are also elementary.

Open questions about theories of operator algebras

- There are continuum many different theories of tracial von Neumann algebras and C*-algebras.
- There are three (to my knowledge) distinct theories of II_1 factors: not Gamma, Gamma but not McDuff and McDuff.
- This cannot be all there is! Conjecture: There are continuum many different theories of II_1 factors.
- Specific questions: are the theories of $L(F_n)$ and $\prod_{n \in N} M_n(C)/U$ distinct?
- Are the theories of $L(F_n)$ and $L(F_m)$ distinct?
- Are the theories of $\prod_{n \in N} M_n(C)/U$ and $\prod_{n \in N} M_n(C)/V$ for different U and V?

Connes' embedding problem

- Does every separable II_1 factor embed into $\mathcal{R}^\omega$?
- General fact: If $A \subseteq B$ then $Th_\forall(B) \subseteq Th_\forall(A)$.
- $\mathcal{R} \hookrightarrow A$ for any II_1 factor so $Th_\forall(A) \subseteq Th_\forall(\mathcal{R})$.
- $\mathcal{R} \prec \mathcal{R}^\omega$ so $Th_\forall(\mathcal{R}) = Th_\forall(\mathcal{R}^\omega)$. It follows then that CEP holds iff $Th_\forall(A) = Th_\forall(\mathcal{R})$ for all II_1 factors A.
- Fact: $Th_\forall(A) = Th_\forall(B)$ iff $Th_\exists(A) = Th_\exists(B)$.
- It is immediate that CEP holds iff the microstate conjecture is true i.e. For any II_1 factor A, $\epsilon > 0$, *-polynomials $p_1(\bar{x}), \ldots, p_n(\bar{x})$ and $\bar{a} \in A$ there is $\bar{b} \in \mathcal{R}$ (alternatively, there is N and $\bar{b} \in M_N$) such that for all $i = 1, \ldots, n$,

$$|tr(p_i(\bar{a})) - tr(p_i(\bar{b}))| \leq \epsilon$$

Even without CEP

- $Th_\forall(\mathcal{R})$ is maximal among universal theories of II_1 factors; it follows by Łoś' theorem that there is a minimal universal theory i.e. there is a separable II_1 factor $\mathcal{S}$ such that for all II_1 factors A, $Th_\forall(\mathcal{S}) \subseteq Th_\forall(A)$.
- Again, it is immediate that for any separable II_1 factor A, $A \hookrightarrow \mathcal{S}^\omega$ (a poor man's resolution to CEP).
- Note: $Th_\forall(\mathcal{S}) = Th_\forall(\mathcal{R})$ iff CEP holds.
- Good question: what could $\mathcal{S}$ look like?

Quantifier complexity

- Any two embeddings of $\mathcal{R}$ into $\mathcal{R}^\omega$ are unitarily equivalent.
- The diagonal embedding of $\mathcal{R}$ into $\mathcal{R}^\omega$ is elementary so any embedding of $\mathcal{R}$ into any model of $Th(\mathcal{R})$ is elementary ($\mathcal{R}$ is a prime model of its theory).
- One typical reason model theoretically for this behaviour is that the given theory has quantifier elimination i.e. for any formula $\varphi(\bar{x})$ and $\epsilon > 0$ there is a quantifier-free formula $\psi(\bar{x})$ such that

$$\sup_{\bar{x} \in B_1} |\varphi(\bar{x}) - \psi(\bar{x})| \leq \epsilon$$

  is part of the theory.
- So, does $Th(\mathcal{R})$ have quantifier elimination?

Quantifier complexity, cont'd

- No! A paper of Nate Brown's contains the following calculation:
- In fact, with a little more work we show that the theory of tracial von Neumann algebras does not have a model companion - it had been conjectured that $Th(\mathcal{R})$ was such.
- Last straw: maybe $Th(\mathcal{R})$ is model complete - this would show that every formula is approximated by sup formulas.

Theorem (Goldbring, H., Sinclair)

If $Th(\mathcal{R})$ is model complete then CEP fails!

Some extra remarks

- The property being exploited here is: for a separable A, any embedding of A into A^ω is elementary. If $Th(A)$ is model complete then A has this property.
- Any UHF algebra has this property as does any strongly self-absorbing algebra (for instance $\mathcal{Z}$, O_2, O_∞).
- Is the theory of any of these algebras model complete?

Stability

We say that the theory of a separable algebra A is stable if for any two ultrafilters U and V on N, A^U and A^V are necessarily isomorphic (independent of the size of the continuum).

Theorem (Farah, H., Sherman)

- *No infinite dimensional C*-algebra is stable.*
- *A separable tracial von Neumann algebra is stable iff it is type 1.*

Theorem (FHS, assume $\neg$ CH)

- *For an infinite dimensional C*-algebra A, there ultrafilters U and V such that* $A' \cap A^U \ncong A' \cap A^V$.
- *For a* II_1 *factor A, A is McDuff iff there ultrafilters U and V such that* $A' \cap A^U \ncong A' \cap A^V$.

Omitting types

- Certain properties are not elementary: UHF, AF, nuclear - how does one recognize these classes of algebras model theoretically?
- Suppose $\mathcal{F}$ is a set of formulas in the variables $\bar{x}$. A type p is a function from $\mathcal{F}$ to R.
- We say that $\bar{a} \in A$ realizes p if $p(\varphi) = \varphi(\bar{a})$ for every $\varphi \in \mathcal{F}$. If there is no such $\bar{a} \in A$, we say A omits p.

Omitting types, cont'd

Claim

There is a countable collection Γ of types such that a C-algebra A is UHF iff A omits all types in Γ.*

Claim

There is a countable collection Γ of types such that a C-algebra A is AF iff A omits all types in Γ.*

We (Ilijas and BH) believe that nuclear and finite nuclear dimension is also a matter of omitting types.

On beyond continuous model theory

- “What do you need the trace for?” (D. Shlyakhtenko)
- Good question!
- On the face of it, we wanted the class we were dealing with to be a class of metric structures - for C*-algebras, that gave us everything but for von Neumann algebras, we only got those with finite trace.
- There exists a natural definition of ultraproduct on von Neumann algebras (due to Yves-Raynaud) and the class is closed under a reasonable notion of subalgebra so ...
- The class of von Neumann algebras should be a CAT (compact abstract theory) - this is a very general framework due to Ben Ya’acov in which much general model theory can be carried out.