Continuous Model Theory

Lecture 2: Discrete vs. continuous; compare and contrast

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Theories

- For a language $\mathcal{L}$, $\text{Sent}_{\mathcal{L}}$ is the set of sentences of $\mathcal{L}$.
- The theory of an $\mathcal{L}$-structure $\mathcal{M}$ is the function $\text{Th}(\mathcal{M}) : \text{Sent}_{\mathcal{L}} \to \mathbb{R}$ defined by, for any sentence φ,

  $$\text{Th}(\mathcal{M})(\varphi) = \varphi^{\mathcal{M}}$$

  Notice that $\text{Th}(\mathcal{M})$ is a linear functional on the space of sentences and is in fact determined by its kernel. We then sometimes refer to $\{\varphi \in \text{Sent}_{\mathcal{L}} : \varphi^{\mathcal{M}} = 0\}$ as the theory of $\mathcal{M}$.
- An ($\mathcal{L}$-)theory is a set of sentences T which is contained in $\text{Th}(\mathcal{M})$ for some $\mathcal{M}$.

Example

We will write out and interpret some formulas and sentences about Hilbert spaces.

- There are universal (sup) sentences expressing the fact that we have a complex inner product space. For instance, we have

$$\sup_{x \in B_1} \sup_{y \in B_1} d_{B_2}(x +_{1,1} y, y +_{1,1} x)$$

  which evaluates to 0 and partially expresses that + is commutative.

- We have the relationship between the inner product and the metric:

$$\sup_{x \in B_n} \sup_{y \in B_n} (d_{B_n}(x, y)^2 - re(\langle x - y, x - y \rangle)).$$

- We also have $\sup_{x \in B_1}(d(x, 0) \dot{-} 1)$.

Example, cont'd

- Consider the sentence, for every $n \in \mathbb{N}$,

$$\sup_{x \in B_n} \min\{1 \dot{-} d(x, 0), \inf_{y \in B_1} d(x, i(y))\}$$

- There are sentences expressing that a Hilbert space is infinite-dimensional (exercise).
- All the sentences we have written out or alluded to specify the theory of infinite-dimensional Hilbert spaces.
- Not too surprisingly it has exactly one separable model up to isomorphism - this is called being separably categorical.
- By an argument which only makes sense later this lecture but very similar to the discrete case, these sentences determine the theory of any model i.e. the theory is complete.

Łoś' Theorem

Theorem

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I, \mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Satisfiability

Definition

- We say a set of sentences Σ in a language $\mathcal{L}$ is satisfied if there is an $\mathcal{L}$-structure $\mathcal{M}$ such that for every sentence in Σ holds in $\mathcal{M}$ i.e. for every $\varphi \in \Sigma$, $\varphi^{\mathcal{M}} = 0$.
- We say such a Σ is finitely satisfied if every finite subset of Σ is satisfied.
- For a set of sentence Σ and $\epsilon > 0$, the ϵ-approximation of Σ is

$$\{|\varphi| \dot{-} \epsilon : \varphi \in \Sigma\}$$

- Σ is approximately finitely satisfied if for every $\epsilon > 0$, the ϵ-approximation of Σ is finitely satisfiable.

Compactness

Theorem

TFAE for a set of sentences Σ in a language $\mathcal{L}$

- Σ *is satisfiable.*
- Σ *is finitely satisfiable.*
- Σ *is approximately finitely satisfiable.*

A metric on formulas

Fix a language $\mathcal{L}$ and fix a tuple of variables $\overline{x}$ from a sequence of sorts $\overline{S}$. We define a pseudo-metric on the formulas with free variables $\overline{x}$ as follows: we define the distance between $\varphi(\overline{x})$ and $\psi(\overline{x})$ to be

$$\sup\{|\varphi^{\mathcal{M}}(\overline{a}) - \psi^{\mathcal{M}}(\overline{a})| : \mathcal{M}, \text{ an } \mathcal{L}\text{-structure, and } \overline{a} \in \mathcal{M}\}$$

We will call this space $\mathcal{F}_{\overline{S}}$. This can also be relativized to all structures satisfying a fixed theory.
Exercise: Check that this is a pseudo-metric on the set of formulas in the free variables $\overline{x}$.

Density character

Definition

We say that the density character of a topological space X is the infinum of the cardinality of a dense subset of X. We will write $\chi(X)$ for the density character of X.

Note: An infinite separable space has countable density character.

Proposition

If $\mathcal{L}$ is countable i.e. there are only countably many relation and function symbols, then for any tuple of sorts $\overline{S}$, $\mathcal{F}_{\overline{S}}$ is separable.

Notation: $\chi(\mathcal{L})$ will mean $\sum_{\overline{S}} \chi(\mathcal{F}_{\overline{S}})$.

Embeddings and elementary submodels

- Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that the universe of $\mathcal{M}$ is a closed subset of $\mathcal{N}$. $\mathcal{M}$ is called a submodel if all functions and relations from $\mathcal{L}$ on $\mathcal{M}$ are the restriction of those from $\mathcal{N}$. We write $\mathcal{M} \subseteq \mathcal{N}$.
- For $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is an *elementary* submodel if, for every $\mathcal{L}$-formula $\varphi(\overline{x})$ and every $\overline{a} \in \mathcal{M}$, $\varphi^{\mathcal{M}}(\overline{a}) = \varphi^{\mathcal{N}}(\overline{a})$. We write $\mathcal{M} \prec \mathcal{N}$.
- An embedding between metric structures is a map which preserves the functions and relations. An embedding is elementary if its image is an elementary submodel of the range.
- For a theory T, $\text{Mod}(T)$ is the category of models of T with elementary maps as morphisms. Such a class is called an elementary class.

Notice that by Łoś' Theorem, any metric structure $\mathcal{M}$ embeds elementarily into its ultrapower $\mathcal{M}^{\mathcal{U}}$ for any ultrafilter $\mathcal{U}$ via the diagonal embedding.

Downward Löwenheim-Skolem

Proposition (Tarski-Vaught)

If $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M}$ is an elementary submodel if for every formula $\varphi(x, \overline{y})$, $r \in \mathbb{R}$ and $\overline{a} \in \mathcal{M}$, if $(\inf_x \varphi(x, \overline{a}))^{\mathcal{N}} < r$ then there is $b \in \mathcal{M}$ such that $(\varphi(b, \overline{a}))^{\mathcal{N}} < r$.

Theorem (DLS)

Suppose that $\mathcal{N}$ is an $\mathcal{L}$-structure and A is a subset of $\mathcal{N}$. Then there is an elementary submodel $\mathcal{M} \subseteq \mathcal{N}$ such that

1. *A is contained in $\mathcal{M}$ and*
2. *for every sort S,*

$$\chi(S^{\mathcal{M}}) \leq \chi(\mathcal{L}) + \chi(A)$$

Some abstract model theory

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

1. *$\mathcal{C}$ is an elementary class.*
2. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.*
3. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Theorem

Continuous first order logic is the maximal logic on metric structures which satisfies compactness, the downward Löwenheim-Skolem theorem and unions of elementary chains.

Types

Fix a theory T in a language $\mathcal{L}$. We consider (partial) functions p on the space of formulas $\mathcal{F}_{\overline{S}}$ for a tuple of sorts $\overline{S}$ to $\mathbb{R}$.

Definition

1. p is a (partial) type if there is a model $\mathcal{M}$ of T and $\overline{a} \in \mathcal{M}$ of the appropriate sort such that $p(\varphi) = \varphi^{\mathcal{M}}(\overline{a})$ for all $\varphi \in \mathrm{dom}(p)$. We say that $\overline{a}$ realizes p.
2. p is called a complete type if the domain of p is $\mathcal{F}_{\overline{S}}$.

Fact

- *p is a type iff it is finitely satisfied i.e. if the restriction to every finite subset of its domain is a type.*
- *A complete type is a linear functional on $\mathcal{F}_{\overline{S}}$.*

A topology on the type space

We fix a language $\mathcal{L}$ and a complete theory T in this language. For a tuple of sorts $\overline{S}$ from $\mathcal{L}$, we define the set $S_{\overline{S}}(T)$ to be all complete types defined on $\mathcal{F}_{\overline{S}}$.
The logic topology on $S_{\overline{S}}(T)$ is the restriction of the weak-* topology on the dual space of $\mathcal{F}_{\overline{S}}$. Equivalently, the collection of sets

$$\{p \in S_{\overline{s}}(T) : p(\varphi) < r\} \text{ for every formula } \varphi \text{ and real number } r,$$

form the collection of basic open sets.

Fact

- *The logic topology on $S_{\overline{s}}(T)$ is compact and Hausdorff.*
- *If φ is a formula then the function f_φ from $S_{\overline{S}}(T)$ to $\mathbb{R}$ given by $p \mapsto p(\varphi)$ is continuous.*

What is a formula?

Proposition

The following are equivalent:

1. *f is a continuous function from $S_{\overline{S}}(T)$ to $\mathbb{R}$.*
2. *f is the uniform limit of functions of the form f_φ i.e. for every n there is a formula φ_n such that for all p, $|f(p) - p(\varphi_n)| \leq 1/n$.*

Definition

A Cauchy sequence of formulas $\overline{\varphi}$ in $\mathcal{F}_{\overline{S}}$ will be called a definable predicate and interpreted in an $\mathcal{L}$-structure $\mathcal{M}$ by

$$\overline{\varphi}^{\mathcal{M}}(\overline{a}) = \lim_{n\to\infty} \varphi_n^{\mathcal{M}}(\overline{a}).$$

Of course what we are doing is extending the notion of formula to the Banach space generated by $\mathcal{F}_{\overline{S}}$.

A metric on the type space

Fix a complete theory T.

- Define a metric on $S_{\overline{S}}(T)$ as follows: for $p, q \in S_{\overline{S}}(T)$, $d(p, q)$ is the infinum of $d^{\mathcal{M}}(\overline{a}, \overline{b})$ where $\mathcal{M}$ ranges over all models of T, $\overline{a} \in \mathcal{M}$ is a realization of p and $\overline{b} \in \mathcal{M}$ is a realization of q. d is computed as the maximum of the values d_S as S ranges over the sorts in $\overline{S}$.
- Claim: d defines a metric on $S_{\overline{S}}(T)$.
- Notice that $d(p, q)$ is always realized - this follows by compactness as does the triangle inequality.

Proposition

The metric topology on $S_{\overline{S}}(T)$ refines the logic topology.

Question: When do the metric and logic topologies coincide?