Continuous Model Theory

Lecture 3: Definability

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Types

Fix a theory T in a language $\mathcal{L}$. We consider (partial) functions p on the space of formulas $\mathcal{F}_{\overline{S}}$ for a tuple of sorts $\overline{S}$ to $\mathbb{R}$.

Definition

1. p is a (partial) type if there is a model $\mathcal{M}$ of T and $\overline{a} \in \mathcal{M}$ of the appropriate sort such that $p(\varphi) = \varphi^{\mathcal{M}}(\overline{a})$ for all $\varphi \in \mathrm{dom}(p)$. We say that $\overline{a}$ realizes p.
2. p is called a complete type if the domain of p is $\mathcal{F}_{\overline{S}}$.

Fact

- *p is a type iff it is finitely satisfied i.e. if the restriction to every finite subset of its domain is a type.*
- *A complete type is a linear functional on $\mathcal{F}_{\overline{S}}$.*

A topology on the type space

We fix a language $\mathcal{L}$ and a complete theory T in this language. For a tuple of sorts $\overline{S}$ from $\mathcal{L}$, we define the set $S_{\overline{S}}(T)$ to be all complete types defined on $\mathcal{F}_{\overline{S}}$.
The logic topology on $S_{\overline{S}}(T)$ is the restriction of the weak-* topology on the dual space of $\mathcal{F}_{\overline{S}}$. Equivalently, the collection of sets

$$\{p \in S_{\overline{s}}(T) : p(\varphi) < r\} \text{ for every formula } \varphi \text{ and real number } r,$$

form the collection of basic open sets.

Fact

- *The logic topology on $S_{\overline{s}}(T)$ is compact and Hausdorff.*
- *If φ is a formula then the function f_φ from $S_{\overline{S}}(T)$ to $\mathbb{R}$ given by $p \mapsto p(\varphi)$ is continuous.*

What is a formula?

Proposition

The following are equivalent:

1. *f is a continuous function from $S_{\overline{S}}(T)$ to $\mathbb{R}$.*
2. *f is the uniform limit of functions of the form f_φ i.e. for every n there is a formula φ_n such that for all p, $|f(p) - p(\varphi_n)| \leq 1/n$.*

Definition

A Cauchy sequence of formulas $\overline{\varphi}$ in $\mathcal{F}_{\overline{S}}$ will be called a definable predicate and interpreted in an $\mathcal{L}$-structure $\mathcal{M}$ by

$$\overline{\varphi}^{\mathcal{M}}(\overline{a}) = \lim_{n\to\infty} \varphi_n^{\mathcal{M}}(\overline{a}).$$

Of course what we are doing is extending the notion of formula to the Banach space generated by $\mathcal{F}_{\overline{S}}$.

A metric on the type space

Fix a complete theory T.

- Define a metric on $S_{\overline{S}}(T)$ as follows: for $p, q \in S_{\overline{S}}(T)$, $d(p, q)$ is the infinum of $d^{\mathcal{M}}(\overline{a}, \overline{b})$ where $\mathcal{M}$ ranges over all models of T, $\overline{a} \in \mathcal{M}$ is a realization of p and $\overline{b} \in \mathcal{M}$ is a realization of q. d is computed as the maximum of the values d_S as S ranges over the sorts in $\overline{S}$.
- Claim: d defines a metric on $S_{\overline{S}}(T)$.
- Notice that $d(p, q)$ is always realized - this follows by compactness as does the triangle inequality.

Proposition

The metric topology on $S_{\overline{S}}(T)$ refines the logic topology.

Question: When do the metric and logic topologies coincide?

A useful lemma

Lemma (MTFMS, 2.10)

Suppose that $F, G : X \to [0,1]$ *are functions such that*

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in X\ (F(x) \le \delta \implies G(x) \le \epsilon)$$

Then there exists an increasing, continuous function $\alpha : [0,1] \to [0,1]$ *such that* $\alpha(0) = 0$ *and*

$$\forall x \in X\ (G(x) \le \alpha(F(x))).$$

Zero sets and distance predicates

Fix a theory T in a language $\mathcal{L}$ and a model $\mathcal{M}$ of T.

- For a definable predicate $\varphi(\bar{x})$, the zero set of φ in $\mathcal{M}$ is

$$\{\bar{a} \in \mathcal{M} : \varphi^{\mathcal{M}}(\bar{a}) = 0\}$$

- If X is a non-empty closed subset of some product of sorts in $\mathcal{M}$ we call $P(x) = d(x, X) = \inf\{d(x, y) : y \in X\}$ the distance predicate for X.

Definable sets

Definition

Suppose we have a theory T in a language $\mathcal{L}$ and S_i for $i \leq n$ are sorts in $\mathcal{L}$. We call an assignment to every model $\mathcal{M}$ of T, a closed subset $X^{\mathcal{M}}$ of $\prod_{j=1}^{m} S_j^{\mathcal{M}}$ a uniform assignment relative to the theory T. This assignment, $\mathcal{M} \mapsto X^{\mathcal{M}}$, is called a *definable set* if, for all formulas $\psi(\bar{x}, \bar{y})$, the functions defined for all $\mathcal{M}$, models of T, by

$$\sup_{\bar{x} \in X^{\mathcal{M}}} \psi^{\mathcal{M}}(\bar{x}, \bar{y}) \quad \text{and} \quad \inf_{\bar{x} \in X^{\mathcal{M}}} \psi^{\mathcal{M}}(\bar{x}, \bar{y})$$

are definable predicates for T.

Critical remarks about definable sets

- A natural source of uniform assignments is the zero-set of any definable predicate.
- If an assignment is a definable set then it is the assignment arising from the zero-set of some definable predicate. Just choose $\psi(\bar{x}, \bar{y}) := d(\bar{x}, \bar{y})$ and parse $\inf_{\bar{x} \in X^{\mathcal{M}}} \psi(\bar{x}, \bar{y})$.
- The definition of definable set could be read

  "Definable sets are those sets you can quantify over."

  Notice in the discrete case, you can quantify over the solution set of any formula.
- There are lots of zero sets which are NOT definable sets.

A second characterization of definable sets

Theorem

Suppose that $\mathcal{M} \mapsto X^{\mathcal{M}}$ is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *The distance predicate $d(\bar{x}, X^{\mathcal{M}})$ is a definable predicate for T.*

Proof.

For (2) implies (1), fix a formula ψ. It is uniformly continuous so using MTFMS 2.10, we can find continuous α such that for all $\bar{x}, \bar{y}$ and $\bar{z}$

$$|\psi(\bar{x}, \bar{z}) - \psi(\bar{y}, \bar{z})| \leq \alpha(d(\bar{x}, \bar{y})).$$

Consider

$$\inf_{\bar{z}}(\psi(\bar{x}, \bar{z}) + \alpha(d(\bar{z}, X))) \text{ and } \inf_{\bar{z} \in X} \psi(\bar{x}, \bar{z}).$$

The claim is that these are equal and the first is a definable predicate. $\square$

A third characterization of definable sets

Theorem

Suppose that $\mathcal{M} \mapsto X^{\mathcal{M}}$ is a uniform assignment relative to a theory T. Then the following are equivalent:

1. *This assignment is a definable set.*
2. *For all sets I, ultrafilters $\mathcal{U}$ on I and families of models of T, $\mathcal{M}_i$ for $i \in I$, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then*

$$X^{\mathcal{M}} = \prod_{\mathcal{U}} X^{\mathcal{M}_i}.$$

Examples of definable sets

- Finite products of sorts; ranges of terms - these are easy because you can clearly quantify over them. The ranges of definable functions are also definable sets.
- The ball of radius 1 around a point in the ball of radius 1 in a Hilbert space. Far more generally, if the underlying metric space has unique geodesics then balls will be definable sets.
- Ultrametrics give examples that are not definable. Here is a toy example: on the interval $[0, 2]$ define the metric d

  $$d(x,y) = \begin{cases} \max\{x,y\} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

  If we fix 0 as a constant than the zero-set of $d(0, x) \leq 1$ doesn't survive ultrapowers. These types of examples arise naturally from metrics associated to certain valuations.
- There are lots of examples from functional analysis but we will leave those for later.

Principal types

Definition

We call a complete type p *principal* if the logic topology on the type space refines the metric topology at p.

Proposition

Suppose that p is a complete type for a complete theory T. The following are equivalent:

1. *p is principal.*
2. *The zero set of p is definable.*
3. *There are formulas φ_m and numbers $\delta_m > 0$ such that for every m, $p(\varphi_m) = 0$ and for any complete type q*

$$\text{if “}\varphi_m(\bar{x}) \leq \delta_m\text{” is in } q \text{ then } d(p,q) \leq \frac{1}{m}.$$

The omitting types theorem

Theorem

Suppose that $\mathcal{L}$ is a separable language, T is a complete theory in $\mathcal{L}$ and p is a complete type. Then every model of T realizes p iff p is principal.

Separable categoricity

Theorem

For a complete theory T in a separable language, T is separably categorical iff all complete types are principal.

Imaginaries: canonical parameters

- Start with a theory T in a language $\mathcal{L}$. The goal is to create the maximal conservative expansion of T. We do this with three separate constructions.
- **Canonical parameters:** If $\varphi(\bar{x}, \bar{y})$ is a formula in $\mathcal{L}$ and $\bar{y}$ is of sort $\bar{S}$ then we add a new sort S_φ with metric symbol d_φ and a function symbol $\pi_\varphi : \bar{S} \to S_\varphi$. We add to T the sentences:

$$\sup_{\bar{y}, \bar{y}'} |d_\varphi(\pi_\varphi(\bar{y}), \pi_\varphi(\bar{y}')) - \sup_{\bar{x}} |\varphi(\bar{x}, \bar{y}) - \varphi(\bar{x}, \bar{y}')| \,|$$

and

$$\sup_z \inf_{\bar{y}} (d_\varphi(\pi_\varphi(\bar{y}), z)).$$

Imaginaries: countable products

Countable products: If $(S_n : n < \omega)$ is a sequence of sorts in $\mathcal{L}$ then we add a new sort S with metric symbol d_S and function symbols $\pi_n : S \to S_n$. Add to T, for all $n < \omega$, the sentences

$$\sup_{x_1 \in S_1} \ldots \sup_{x_n \in S_n} \inf_{y \in S} \max_{i \leq n} d_i(\pi_i(y), x_i)$$

where d_i is the metric on S_i, and

$$\sup_{x,y \in S} |d_S(x,y) - \sum_{i=1}^{n} \frac{d_i(\pi_i(x), \pi_i(y))}{2^i}| \dot{-} \frac{1}{2^n}.$$

Imaginaries: definable sets

Definable sets: If $A(x_1, ..., x_n)$ is a definable set in T then add a sort S_A with metric symbol d_A, and function symbols $f_i : S_A \to S_i$ for $i \leq n$ where S_i is the sort of x_i. We add to T the sentences

$$|A(x_1, ..., x_n) - \inf_y \max_{i \leq n} d_i(x_i, f_i(y))|$$

where d_i is the metric symbol on S_i, and

$$|d_A(x, y) - \max_{i \leq n} d_i(x_i, f_i(y))|.$$

Conceptual completeness

T^{eq} will be the smallest theory expanding T which is closed under canonical parameters, countable products and definable sets. One proves by induction on the construction that this is a conservative expansion of T.

Theorem

If T is a complete theory and T' is a conservative extension of T i.e. the forgetful functor from $Mod(T')$ to $Mod(T)$ is an equivalence of categories then T' can be interpreted in T^{eq}.