Continuous Model Theory

Notes and references

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The goal of these notes is to supplement the tutorial lectures by summarizing several topics and providing references.

Ultraproducts

The history of the ultraproduct is very interesting and more colourful than most model theorists are aware. A short history is contained in [30]. The thumbnail sketch is that an ultraproduct-like construction in the operator theory context was introduced by Kaplansky and Wright in the early 1950's. Sakai also used the ultraproduct in the early 1960's again in the service of operator algebras. McDuff's systematic use in her paper [27] highlighted how important the construction had become in the operator algebra world. In parallel, the seminal introduction of the ultraproduct in model theory by Łoś, [29], and its use by Robinson in the development of non-standard analysis, were key early moments in model theory. It doesn't seem that anyone at that time saw anything more than a formal connection between the two uses of ultraproducts. Keisler, [24], provides a brief history on the model theory side in which he gives the nod for a precursor to the ultraproduct to Skolem and interestingly to Hewitt [22] who was working on rings of continuous functions. Although there was prior work done by Krivine, Stern and others, the first systematic connection between the analytic ultraproduct construction and model theory particularly for Banach spaces was the work of Henson [19] and the introduction of positive bounded logic. A general exposition appears with Iovino in [21].

A short history of continuous model theory

The idea of a logic which took real numbers as truth values goes back to Łukasiewicz and Tarski at least in the propositional setting. The modern understanding of continuous model theory had several antecedents. Chang and Keisler wrote a book [11] in which they consider logics with truth values in arbitrary compact Hausdorff spaces. They had many quantifiers and the logic did not catch on. As mentioned above, Henson introduced positive bounded logic mostly in the service of considering the model theory of normed vector spaces with additional structure; see also [20]. Ben Yaacov introduced the very general setting of cats ([3, 4, 5]) in order to capture expansions of first order logic that would allow seamless use of hyperimaginaries, building on work on Robinson theories by Hrushovski, Pillay and others. In particular, it was realized that Hausdorff cats carried a metric and Henson and Ben Yaacov introduced a logic for metric structures which is the subject of this tutorial. It is first presented in [10] which strangely was published after the de facto standard [9]. The abstract model theory of metric structures is discussed in a paper of Iovino [23] in which he proves the Lindström-like characterization of continuous first order logic which is presented in the lectures. Note that this paper pre-dates the definition of metric structure!

Important topics not covered

Quantifier elimination, saturation, stability. One can only get to so much in a short tutorial but the basics of these important topics are covered in both [10] and [9]. The result which links the number of ultrapowers of a separable metric structure to stability first appears in [15] and the more general result is in [17].

Henkin construction. The general Henkin construction is alluded to in [9] and very general details appear in a paper by Farah and Magidor ([16]). They show that there is no simple criteria for determining if a partial type is omissible. Another treatment of the Henkin construction appears in [13].

Unbounded metric structures. As we saw, requiring a bound on the metric in general applications can be annoying. The logic of unbounded metric structures was worked out with two different formalisms by Ben Yaacov in [6] and Luther [26].

Definable sets

Definable sets and their role in continuous model theory is of course discussed thoroughly in [9] and [10]. Additional details particularly about definable functions appear in [7]. The general notion first appears in discussions by Henson of separable categoricity and the role of principal types. The treatment here follows [13] where the connection to the Beth definability theorem appears (not discussed in the lectures). The notion of weakly stable relations in the context of C*-algebras appears in the work of Loring [25].

Imaginaries and conceptual completeness

Imaginaries are discussed in both [9] and [10]. They also appear in [1] and [13]. A proof of conceptual completeness appears in [18] and the forthcoming [2].

Urysohn space

The construction of Urysohn space goes back to Urysohn in 1924 just before his death. The model theory of Urysohn space in the continuous setting is worked out by Usvyatsov in [31]. The details of the Fraïssé construction in the continuous setting appears in [8] but see also [28]. For a variant that is relevant to C*-algebras see also [12].

C*-algebras

The basic model theory of C*-algebras was worked out in [15] which also treats the case of general tracial von Neumann algebras and II_1 factors. An answer to the McDuff question discussed on the lecture slides appears in [14]. The current state of the art in the model theory of C*-algebras is [13] that contains references to almost all of the existing literature.

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