1. (5 marks) Put your answer in the space provided for each part.

   (a) The range of a linear transformation is a vector space. True or False.

   T

   (b) The real vector space $\mathbf{R}^3$ and P_2, real polynomials of degree ≤ 2 are isomorphic. True or False.

   T

   (c) Suppose that $T : V \to W$ is a linear transformation, $\dim(V) = 6$, $\dim(W) = 3$ and $\text{nullity}(T) = 4$. What is the rank of T?

   2

   (d) If V is an inner product space and W is a finite-dimensional subspace not equal to V then the projection from V to W is one-to-one. True or False.

   F

   (e) The set $\{1, \sin(x), \sin(2x), \sin(3x, \sin(4x)\}$ is a linearly independent subset of $C[0, 2\pi]$ with respect to the inner product $\langle f, g \rangle = \int_0^{2\pi} fg\,dx$. True or False.

   T

continued ...

2. **(5 marks) In the inner product space of continuous functions on [-1,1] with the inner product given by**

$$\langle f, g\rangle = \int_{-1}^{1} fg\,dx$$

find the projection of e^x onto the subspace generated by 1 and x.

Let W be the subspace generated by 1 and x. Notice that $\langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0$ so 1 and x form an orth. basis.

$\|1\|^2 = 2$, $\|x\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$.

$$\text{proj}_1 e^x = \frac{\langle e^x, 1\rangle}{\|1\|^2} 1 = \frac{\int_{-1}^{1} e^x dx}{2} = \frac{e - e^{-1}}{2}$$

$$\text{proj}_x e^x = \frac{\langle e^x, x\rangle}{\|x\|^2} x = \frac{\int_{-1}^{1} x e^x dx}{\left(\frac{2}{3}\right)} x.$$

$u = x \quad v = e^x$

$du = dx \quad dv = e^x dx$

$\int x e^x dx = x e^x - e^x + C$

$= e^x(x-1) + C.$

But $\int_{-1}^{1} x e^x dx = e^x(x-1)\Big|_{-1}^{1} = 2e^{-1}$

so $\text{proj}_x e^x = \frac{2e^{-1}}{\left(\frac{2}{3}\right)} x = 3e^{-1} x$

So $\text{proj}_W e^x = \frac{e - e^{-1}}{2} + 3e^{-1} x$.

continued ...

3. Let V be the inner product space of continuous functions on $[0, 2\pi]$ with inner product given by

$$\langle f, g \rangle = \int_0^{2\pi} fg \, dx.$$

(a) (3 marks) Compute the projection of x onto $\sin(2x)$ and $\cos(2x)$ in this inner product space.

$$\text{proj}_{\sin(2x)} x = \frac{\int_0^{2\pi} x \sin 2x \, dx}{\|\sin 2x\|^2} \sin(2x) = \frac{-\frac{x\cos 2x}{2}\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \cos 2x \, dx}{\pi} \sin 2x \qquad u = x \quad du = dx \quad v = \frac{-\cos 2x}{2} \quad dv = \sin 2x \, dx$$

$$= -\frac{2\pi}{2\pi} \sin 2x = -\sin 2x.$$

$$\text{proj}_{\cos 2x} x = \frac{\int_0^{2\pi} x \cos 2x \, dx}{\|\cos 2x\|^2} \cos 2x = \frac{\frac{x \sin 2x}{2}\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \sin 2x \, dx}{\pi} \cos 2x = 0.$$

(b) (2 marks) If the Fourier series for the function $f(x) = x^2$ is

$$\frac{4\pi}{3} - 4\pi\left(\sin(x) + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \dots\right) + 4\left(\cos(x) + \frac{1}{4}\cos(2x) + \frac{1}{9}\cos(3x)\dots\right)$$

what is the Fourier series for $2 - x^2$?

$$\left(2 - \frac{4\pi}{3}\right) + 4\pi\left(\sin(x) + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right)$$

$$- 4\left(\cos(x) + \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x + \dots\right)$$

continued ...

4. Suppose that $v_1 = (2,1)$ and $v_2 = (1,1)$ and that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation such that $T(v_1) = (0,1)$ and $T(v_2) = (1,0)$.

   (a) (2 marks) Compute $T(0,1)$. $2v_2 - v_1 = (0,1)$ so

$$T(0,1) = 2T(v_2) - T(v_1)$$
$$= (2,0) - (0,1) = (2,-1).$$

   (b) (3 marks) Write an expression for $T(x,y)$. We need to write (x,y) in terms of v_1 and v_2. If $c_1 v_1 + c_2 v_2 = (x,y)$ then

$$(2c_1 + c_2, c_1 + c_2) = (x,y) \text{ or}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ so } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} x-y \\ 2y-x \end{pmatrix}.$$

$$T(x,y) = T\big((x-y)v_1 + (2y-x)v_2\big) = (x-y)(0,1) + (2y-x)(1,0)$$
$$= (2y-x, x-y).$$

continued ...

5. (a) (2 marks) Suppose that V and W are vector spaces and that $T : V \to W$. Define what it means to say that T is a linear transformation.

T is a linear transformation if

1) for all $v_1, v_2 \in V$, $T(v_1 + v_2) = T(v_1) + T(v_2)$ and

2) for all scalars c and $v \in V$, $T(cv) = cT(v)$.

(b) (3 marks) Suppose that U, V and W are vector spaces and $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations. Prove that $T_2 \circ T_1$ is a linear transformation.

We need to check the two conditions above:

1) If $u_1, u_2 \in U$ then

$$
\begin{aligned}
(T_2 \circ T_1)(u_1 + u_2) &= T_2(T_1(u_1+u_2)) \\
&= T_2(T_1(u_1) + T_1(u_2)) && \text{since } T_1 \text{ is a lin. trans.} \\
&= T_2(T_1(u_1)) + T_2(T_1(u_2)) && \text{since } T_2 \text{ is a lin trans.} \\
&= (T_2 \circ T_1)(u_1) + (T_2 \circ T_1)(u_2).
\end{aligned}
$$

2) If c is a scalar and $u \in U$ then

$$
\begin{aligned}
(T_2 \circ T_1)(cu) = T_2(T_1(cu)) &= T_2(cT_1(u)) \\
&= c\,T_2(T_1(u)) \\
&= c\,(T_2 \circ T_1)(u).
\end{aligned}
$$

THE END