Complex inner product spaces

Definition

An inner product on a complex vector space V is a function that associates a complex number $\langle u, v\rangle$ to each pair of vectors $u, v \in V$ such that the following axioms are satisfied, for every u, v and w in V, and scalar k:

1. $\langle u, v\rangle = \overline{\langle v, u\rangle}$,
2. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$,
3. $\langle (ku), v\rangle = k\langle u, v\rangle$, and
4. $\langle u, u\rangle \geq 0$. Moreover $\langle u, u\rangle = 0$ iff $u = 0$.

V together with an inner product is called a complex inner product space.

The definition of norm and distance in a complex inner product space is the same as in the real case.

Orthogonal complement

Definition

1. If u and v are vectors in an inner product space then we say that u and v are orthogonal if $\langle u, v\rangle = 0$.
2. If W is a subspace of an inner product space V then we say that $v \in V$ is orthogonal to W if v is orthogonal to every $w \in W$. The set of all $v \in V$ which are orthogonal to W is called the orthogonal complement of W and is written $W^\perp$.

Theorem

If W is a subspace of an inner product space V then

1. *$W^\perp$ is a subspace of V,*
2. *W and $W^\perp$ have only 0 in their intersection, and*
3. *if V is finite-dimensional then $(W^\perp)^\perp = W$.*

Connection with matrices

Theorem

Suppose that A is any $m \times n$ matrix. Then

1. *the nullspace of A and the row space of A are orthogonal complements in R^n with respect to the usual (Euclidean) inner product on R^n.*
2. *the nullspace of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product on R^m.*

Orthonormal bases

Definition

A set S of non-zero vectors in an inner product space is called orthogonal if every distinct pair of vectors in S is orthogonal. S is called orthonormal if it is orthogonal and every vector has length one.

Theorem (6.3.1)

If S is an orthogonal set of non-zero vectors in an inner product space then S is linearly independent.

Properties of orthonormal bases

Theorem (6.3.2)

(a) If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis for an inner product space V then for every $u \in V$,

$$u = \frac{\langle u, v_1 \rangle}{\|v_1\|} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|} + \ldots + \frac{\langle u, v_n \rangle}{\|v_n\|} v_n$$

(b)If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space V then for every $u \in V$,

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \ldots + \langle u, v_n \rangle v_n$$

Properties of orthonormal bases, cont'd

Theorem

If S is an orthonormal basis for an n-dimensional inner product space V and then for every $u, v \in V$, if

$$u = (u_1, u_2, \ldots, u_n)_S \text{ and } v = (v_1, v_2, \ldots, v_n)_S$$

then

1. $||u|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$
2. $d(u, v) = \sqrt{(u_1 - v_1)^2 + \ldots (u_n - v_n)^2}$
3. $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$